12. **(a)** Note that $x^2 y = 1125$, so $y = \frac{1125}{x^2}$. Then

$$\begin{aligned} c &= 5(x^2 + 4xy) + 10xy \\ &= 5x^2 + 30xy \\ &= 5x^2 + 30x\left(\frac{1125}{x^2}\right) \\ &= 5x^2 + 33,750x^{-1} \end{aligned}$$

$$\frac{dc}{dx} = 10x - 33,750x^{-2} = \frac{10(x^3 - 3375)}{x^2}$$

The critical point occurs at $x = 15$. Since $\frac{dc}{dx} < 0$ for $0 < x < 15$ and $\frac{dc}{dx} > 0$ for $x > 15$, the critical point corresponds to the minimum cost. The values of x and y are $x = 15$ ft and $y = 5$ ft.

(b) The material for the tank costs 5 dollars/sq ft and the excavation charge is 10 dollars for each square foot of the cross-sectional area of one wall of the hole.

13. Let x be the height in inches of the printed area. Then the width of the printed area is $\frac{50}{x}$ in. and the overall dimensions are $x + 8$ in. by $\frac{50}{x} + 4$ in. The amount of paper used is

$$A(x) = (x+8)\left(\frac{50}{x} + 4\right) = 4x + 82 + \frac{400}{x} \text{ in}^2.$$

Then $A'(x) = 4 - 400x^{-2} = \frac{4(x^2 - 100)}{x^2}$ and the critical point (for $x > 0$) occurs at $x = 10$. Since $A'(x) < 0$ for $0 < x < 10$ and $A'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of paper. Using $x + 8$ and $\frac{50}{x} + 4$ for $x = 10$, the overall dimensions are 18 in. high by 9 in. wide.

14. **(a)** $s(t) = -16t^2 + 96t + 112$
$v(t) = s'(t) = -32t + 96$
At $t = 0$, the velocity is $v(0) = 96$ ft/sec.

(b) The maximum height occurs when $v(t) = 0$, when $t = 3$. The maximum height is $s(3) = 256$ ft and it occurs at $t = 3$ sec.

(c) Note that $s(t) = -16t^2 + 96t + 112$
$= -16(t+1)(t-7)$,
so $s = 0$ at $t = -1$ or $t = 7$. Choosing the positive value, of t, the velocity when $s = 0$ is $v(7) = -128$ ft/sec.

15. We assume that a and b are held constant. Then $A(\theta) = \frac{1}{2}ab \sin\theta$ and $A'(\theta) = \frac{1}{2}ab\cos\theta$. The critical point (for $0 < \theta < \pi$) occurs at $\theta = \frac{\pi}{2}$. Since $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{2}$ and $A'(\theta) < 0$ for $\frac{\pi}{2} < \theta < \pi$, the critical point corresponds to the maximum area. The angle that maximizes the triangle's area is $\theta = \frac{\pi}{2}$ (or 90°).

16. Let the can have radius r cm and height h cm. Then $\pi r^2 h = 1000$, so $h = \frac{1000}{\pi r^2}$. The area of material used is

$A = \pi r^2 + 2\pi rh = \pi r^2 + \frac{2000}{r}$, so

$\frac{dA}{dr} = 2\pi r - 2000r^{-2} = \frac{2\pi r^3 - 2000}{r^2}$. The critical point occurs at

$r = \sqrt[3]{\frac{1000}{\pi}} = 10\pi^{-1/3}$ cm. Since $\frac{dA}{dr} < 0$ for $0 < r < 10\pi^{-1/3}$ and $\frac{dA}{dr} > 0$ for $r > 10\pi^{1/3}$, the critical point corresponds to the least amount of material used and hence the lightest possible can. The dimensions are $r = 10\pi^{-1/3} \approx 6.83$ cm and $h = 10\pi^{-1/3} \approx 6.828$ cm. In Example 4, because of the top of the can, the "best" design is less big around and taller.

17. Note that $\pi r^2 h = 1000$, so $h = \frac{1000}{\pi r^2}$. Then

$A = 8r^2 + 2\pi rh = 8r^2 + \frac{2000}{r}$, so

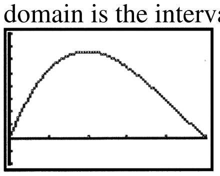
$\frac{dA}{dr} = 16r - 2000r^{-2} = \frac{16(r^3 - 125)}{r^2}$. The critical point occurs at $r = \sqrt[3]{125} = 5$ cm. Since

$\frac{dA}{dr} < 0$ for $0 < r < 5$ and $\frac{dA}{dr} > 0$ for $r > 5$, the critical point corresponds to the least amount of aluminum used or wasted and hence the most economical can. The dimensions are $r = 5$ cm and $h = \frac{40}{\pi}$, so the ratio of h to r is $\frac{8}{\pi}$ to 1.

18. **(a)** The base measures $10 - 2x$ in. by $\frac{15-2x}{2}$ in, so the volume formula is

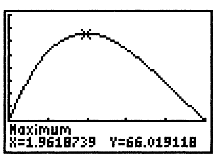
$$V(x) = \frac{x(10-2x)(15-2x)}{2}$$
$$= 2x^3 - 25x^2 + 75x.$$

(b) We require $x > 0$, $2x < 10$, and $2x < 15$. Combining these requirements, the domain is the interval (0, 5).



[0, 5] by [−20, 80]

(c)



[0, 5] by [−20, 80]

The maximum volume is approximately 66.02 in^3 when $x \approx 1.96$ in.

(d) $V'(x) = 6x^2 - 50x + 75$

The critical point occurs when $V'(x) = 0$,

at $x = \frac{50 \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)}$

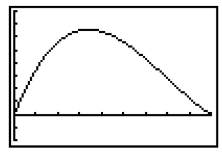
$$= \frac{50 \pm \sqrt{700}}{12}$$
$$= \frac{25 \pm 5\sqrt{7}}{6},$$

that is, $x \approx 1.96$ or $x \approx 6.37$. We discard the larger value because it is not in the domain. Since $V''(x) = 12x - 50$, which is negative when $x \approx 1.96$, the critical point corresponds to the maximum volume. The maximum volume occurs when $x = \frac{25 - 5\sqrt{7}}{6} \approx 1.96$, which confirms the result in (c).

19. **(a)** The "sides" of the suitcase will measure $24 - 2x$ in. by $18 - 2x$ in. and will be $2x$ in. apart, so the volume formula is

$$V(x) = 2x(24-2x)(18-2x)$$
$$= 8x^3 - 168x^2 + 864x.$$

(b) We require $x > 0$, $2x < 18$, and $2x < 24$. Combining these requirements, the domain is the interval (0, 9).

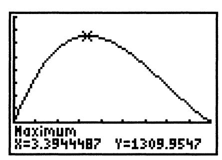


[0, 9] by [−400, 1600]

(c) $V'(x) = 24x^2 - 336x + 864$
$$= 24(x^2 - 14x + 36)$$

The critical point is at $x = 7 \pm \sqrt{13}$, that is, $x \approx 3.39$ or $x \approx 10.61$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 14)$, which is negative when $x \approx 3.39$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = 7 - \sqrt{13} \approx 3.39$.

(d)



[0, 9] by [−400, 1600]

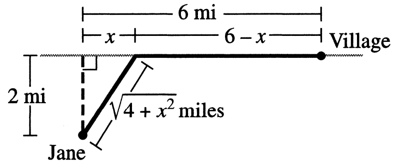
The maximum volume is approximately 1309.95 in^3 when $x \approx 3.39$ in.

(e)
$$8x^3 - 168x^2 + 864x = 1120$$
$$8(x^3 - 21x^2 + 108x - 140) = 0$$
$$8(x-2)(x-5)(x-14) = 0$$

Since 14 is not in the domain, the possible values of x are $x = 2$ in. or $x = 5$ in.

(f) The dimensions of the resulting box are $2x$ in., $(24 - 2x)$ in., and $(18 - 2x)$ in. Each of these measurements must be positive, so that gives the domain of (0, 9)

20.



Let x be the distance from the point on the shoreline nearest Jane's boat to the point where she lands her boat. Then she needs to row $\sqrt{4+x^2}$ mi at 2 mph and walk $6 - x$ mi at 5 mph. The total amount of time to reach the village is $f(x) = \frac{\sqrt{4+x^2}}{2} + \frac{6-x}{5}$ hours ($0 \le x \le 6$). Then $f'(x) = \frac{1}{2}\frac{1}{2\sqrt{4+x^2}}(2x) - \frac{1}{5} = \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5}$.

Solving $f'(x) = 0,$ we have:

$$\begin{aligned} \frac{x}{2\sqrt{4+x^2}} &= \frac{1}{5} \\ 5x &= 2\sqrt{4+x^2} \\ 25x^2 &= 4(4+x^2) \\ 21x^2 &= 16 \\ x &= \pm\frac{4}{\sqrt{21}} \end{aligned}$$

We discard the negative value of x because it is not in the domain. Checking the endpoints and critical point, we have $f(0) = 2.2, f\left(\frac{4}{\sqrt{21}}\right) \approx 2.12$, and $f(6) \approx 3.16.$ Jane should land her boat $\frac{4}{\sqrt{21}} \approx 0.87$ miles down the shoreline from the point nearest her boat.

21. If the upper right vertex of the rectangle is located at $(x, 4\cos 0.5x)$ for $0 < x < \pi$, then the rectangle has width $2x$ and height $4\cos 0.5x$, so the area is $A(x) = 8x\cos 0.5x$.
Then

$$\begin{aligned} A'(x) &= 8x(-0.5\sin 0.5x) + 8(\cos 0.5x)(1) \\ &= -4x\sin 0.5x + 8\cos 0.5x. \end{aligned}$$

Solving $A'(x)$ graphically for $0 < x < \pi$, we find that $x \approx 1.72$. Evaluating $2x$ and $4\cos 0.5x$ for $x \approx 1.72$, the dimensions of the rectangle are approximately 3.44 (width) by 2.61 (height), and the maximum area is approximately 8.98.

22. Let the radius of the cylinder be r cm, $0 < r < 10$. Then the height is $2\sqrt{100-r^2}$ and the volume is $V(r) = 2\pi r^2\sqrt{100-r^2}\,\text{cm}^3$. Then

$$\begin{aligned} V'(r) &= 2\pi r^2\left(\frac{1}{2\sqrt{100-r^2}}\right)(-2r) + (2\pi\sqrt{100-r^2})(2r) \\ &= \frac{-2\pi r^3 + 4\pi r(100-r^2)}{\sqrt{100-r^2}} \\ &= \frac{2\pi r(200-3r^2)}{\sqrt{100-r^2}} \end{aligned}$$

The critical point for $0 < r < 10$ occurs at $r = \sqrt{\frac{200}{3}} = 10\sqrt{\frac{2}{3}}$. Since $V'(r) > 0$ for $0 < r < 10\sqrt{\frac{2}{3}}$ and $V'(r) > 0$ for $10\sqrt{\frac{2}{3}} < r < 10$, the critical point corresponds to the maximum volume. The dimensions are $r = 10\sqrt{\frac{2}{3}} \approx 8.16\,\text{cm}$ and $h = \frac{20}{\sqrt{3}} \approx 11.55$ cm, and the volume is $\frac{4000\pi}{3\sqrt{3}} \approx 2418.40\ \text{cm}^3$.

23. Set $r'(x) = c'(x): 4x^{-1/2} = 4x.$ The only positive critical value is $x = 1$, so profit is maximized at a production level of 1000 units. Note that $(r-c)''(x) = -2(x)^{-3/2} - 4 < 0$ for all positive x, so the Second Derivative Test confirms the maximum.

24. Set $r'(x) = c'(x): 2x/(x^2+1)^2 = (x-1)^2$. We solve this equation graphically to find that $x \approx 0.294$. The graph of $y = r(x) - c(x)$ shows a minimum at $x \approx 0.294$ and a maximum at $x \approx 1.525$, so profit is maximized at a production level of about 1,525 units.

25. Set $\bar{c}(x) = \dfrac{c(x)}{x} = x^2 - 10x + 30.$ The only positive solution to $\dfrac{d}{dx}\left(\dfrac{c(x)}{x}\right) = 0$ is $x = 5$, so average cost is minimized at a production level of 5000 units. Note that $\dfrac{d^2}{dx^2}\left(\dfrac{c(x)}{x}\right) = 2 > 0$ for all positive x, so the Second Derivative Test Confirms the minimum.

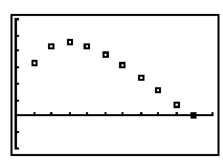
26. Set $\bar{c}(x) = \dfrac{c(x)}{x} = e^x - 2x.$ The only positive solution to $\dfrac{d}{dx}\left(\dfrac{c(x)}{x}\right) = 0$ is $x = \ln 2$, so average cost is minimized at a production level of 1000 ln 2, which is about 693 units. Note that $\dfrac{d^2}{dx^2}\left(\dfrac{c(x)}{x}\right) = e^x > 0$ for all positive x, so the Second Derivative Test confirms the minimum.

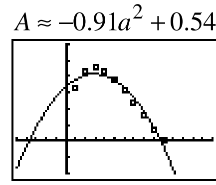
27. Revenue: $r(x) = [200 - 2(x-50)]x$
$= -2x^2 + 300x$
Cost: $c(x) = 6000 + 32x$
Profit: $p(x) = r(x) - c(x)$
$= -2x^2 + 268x - 6000,$
$50 \le x \le 80$
Since $p'(x) = -4x + 268 = -4(x-67)$, the critical point occurs at $x = 67$. This value represents the maximum because $p''(x) = -4$, which is negative for all x in the domain. The maximum profit occurs if 67 people go on the tour.

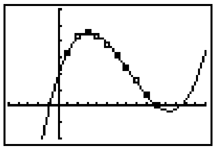
28. **(a)** $f'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x}(1-x)$
The critical point occurs at $x = 1$. Since $f'(x) > 0$ for $0 \le x < 1$ and $f'(x) < 0$ for $x > 1$, the critical point corresponds to the maximum value of f. The absolute maximum of f occurs at $x = 1$.

(b) To find the values of b, use grapher techniques to solve $xe^{-x} = 0.1e^{-0.1}$, $xe^{-x} = 0.2e^{-0.2}$, and so on. To find the values of A, calculate $(b-a)\,ae^{-a}$, using the unrounded values of b. (Use the *list* features of the grapher in order to keep track of the unrounded values for part (d).)

a	b	A
0.1	3.71	0.33
0.2	2.86	0.44
0.3	2.36	0.46
0.4	2.02	0.43
0.5	1.76	0.38
0.6	1.55	0.31
0.7	1.38	0.23
0.8	1.23	0.15
0.9	1.11	0.08
1.0	1.00	0.00

(c)

[0, 1.1] by [–0.2, 0.6]

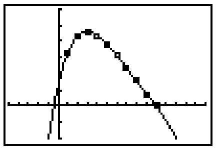
(d) Quadratic:
$A \approx -0.91a^2 + 0.54a + 0.34$

[–0.5, 1.5] by [–0.2, 0.6]
Cubic:
$A \approx 1.74a^3 - 3.78a^2 + 1.86a + 0.19$



[–0.5, 1.5] by [–0.2, 0.6]

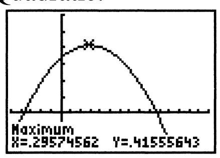
Quartic:

$$A \approx -1.92a^4 + 5.96a^3 - 6.87a^2 + 2.71a + 0.12$$



[–0.5, 1.5] by [–0.2, 0.6]

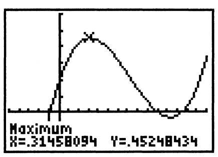
(e) Quadratic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quadratic regression equation, the maximum area occurs at $a \approx 0.30$ and is approximately 0.42.

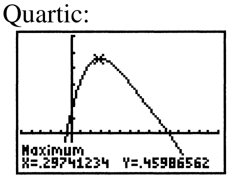
Cubic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the cubic regression equation, the maxiumu area occurs at $a \approx 0.31$ and is approximately 0.45.

Quartic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quartic regression equation the maximum area occurs at $a \approx 0.30$ and is approximately 0.46.

29. (a) $f'(x)$ is a quadratic polynomial, and as such it can have 0, 1, or 2 zeros. If it has 0 or 1 zeros, then its sign never changes, so $f(f(x)$ has no local extrema.

If $f'(x)$ has 2 zeros, then its sign changes twice, and $f(f(x)$ has 2 local extrema at those points.

(b) Possible answers:

No local extrema: $y = x^3$;

2 local extrema: $y = x^3 - 3x$

30. Let x be the length in inches of each edge of the square end, and let y be the length of the box. Then we require $4x + y \le 108$. Since our goal is to maximize volume, we assume $4x + y = 108$ and so $y = 108 - 4x$. The volume is $V(x) = x^2(108 - 4x) = 108x^2 - 4x^3$, where $0 < x < 27$. Then $V' = 216x - 12x^2 = -12x(x - 18)$, so the critical point occurs at $x = 18$ in. Since $V'(x) > 0$ for $0 < x < 18$ and $V'(x) < 0$ for $18 < x < 27$, the critical point corresponds to the maximum volume. The dimensions of the box with the largest possible volume are 18 in. by 18 in. by 36 in.

31. Since $2x + 2y = 36$, we know that $y = 18 - x$. In part (a), the radius is $\dfrac{x}{2\pi}$ and the height is $18 - x$, and so the volume is given by

$$\pi r^2 h = \pi\left(\frac{x}{2\pi}\right)^2 (18 - x) = \frac{1}{4\pi} x^2 (18 - x).$$

In part (b), the radius is x and the height is $18 - x$, and so the volume is given by $\pi r^2 h = \pi x^2 (18 - x)$. Thus, each problem requires us to find the value of x that maximizes $f(x) = x^2(18 - x)$ in the interval $0 < x < 18$, so the two problems have the same answer.

To solve either problem, note that $f(x) = 18x^2 - x^3$ and so $f'(x) = 36x - 3x^2 = -3x(x - 12)$. The critical point occurs at $x = 12$. Since $f'(x) > 0$ for $0 < x < 12$ and $f'(x) < 0$ for $12 < x < 18$, the critical point corresponds to the maximum value of $f(x)$. To maximize the volume in either part (a) or (b), let $x = 12$ cm and $y = 6$ cm.

32. Note that $h^2 + r^2 = 3$ and so $r = \sqrt{3 - h^2}$. Then the volume is given by

$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(3 - h^2)h = \pi h - \frac{\pi}{3} h^3 \text{ for}$$

$$0 < h < \sqrt{3}, \text{ and so } \frac{dV}{dh} = \pi - \pi h^2 = \pi(1 - h^2).$$

The critical point (for $h > 0$) occurs at $h = 1$.

Since $\frac{dV}{dh} > 0$ for $0 < h < 1$ and $\frac{dV}{dh} < 0$ for $1 < h < \sqrt{3}$, the critical point corresponds to the maximum volume. The cone of greatest volume has radius $\sqrt{2}$ m, height 1 m, and volume $\frac{2\pi}{3}\text{m}^3$.

33. **(a)** We require $f(x)$ to have a critical point at $x = 2$. Since $f'(x) = 2x - ax^{-2}$, we have $f'(2) = 4 - \frac{a}{4}$ and so our requirement is that $4 - \frac{a}{4} = 0$. Therefore, $a = 16$. To verify that the critical point corresponds to a local minimum, note that we now have $f'(x) = 2x - 16x^{-2}$ and so $f''(x) = 2 + 32x^{-3}$, so $f''(2) = 6$, which is positive as expected. So, use $a = 16$.

(b) We require $f''(1) = 0$. Since $f'' = 2 + 2ax^{-3}$, we have $f''(1) = 2 + 2a$, so our requirement is that $2 + 2a = 0$. Therefore, $a = -1$. To verify that $x = 1$ is in fact an inflection point, note that we now have $f''(x) = 2 - 2x^{-3}$, which is negative for $0 < x < 1$ and positive for $x > 1$. Therefore, the graph of f is concave down in the interval $(0, 1)$ and concave up in the interval $(1, \infty)$, So, use $a = -1$.

34. $f'(x) = 2x - ax^{-2} = \frac{2x^3 - a}{x^2}$, so the only sign change in $f'(x)$ occurs at $x = \left(\frac{a}{2}\right)^{1/3}$, where the sign changes from negative to positive. This means there is a local minimum at that point, and there are no local maxima.

35. **(a)** Note that $f'(x) = 3x^2 + 2ax + b$. We require $f'(-1) = 0$ and $f'(3) = 0$, which give $3 - 2a + b = 0$ and $27 + 6a + b = 0$. Subtracting the first equation from the second, we have $24 + 8a = 0$ and so $a = -3$.
Substituting into the first equation, we have $9 + b = 0$, so $b = -9$. Therefore, our equation for $f(x)$ is $f(x) = x^3 - 3x^2 - 9x$.
To verify that we have a local maximum at $x = -1$ and a local minimum at $x = 3$, note that
$f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$,
which is positive for $x < -1$, negative for $-1 < x < 3$, and positive for $x > 3$. So, use $a = -3$ and $b = -9$.

(b) Note that $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$. We require $f'(4) = 0$ and $f''(1) = 0$, which give $48 + 8a + b = 0$ and $6 + 2a = 0$. By the second equation, $a = -3$, and so the first equation becomes $48 - 24 + b = 0$. Thus $b = -24$. To verify that we have a local minimum at $x = 4$, and an inflection point *at* $x = 1$, note that we now have $f''(x) = 6x - 6$. Since f'' changes sign at $x = 1$ and is positive at $x = 4$, the desired conditions are satisfied. So, use $a = -3$ and $b = -24$.

36. Refer to the illustration in the problem statement. Since $x^2 + y^2 = 9$, we have $x = \sqrt{9 - y^2}$. Then the volume of the cone is given by

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi x^2 (y+3) \\ &= \frac{1}{3}\pi(9 - y^2)(y+3) \\ &= \frac{\pi}{3}(-y^3 - 3y^2 + 9y + 27), \end{aligned}$$

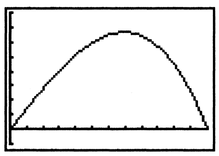
for $-3 < y < 3$.

Thus
$$\begin{aligned} \frac{dV}{dy} &= \frac{\pi}{3}(-3y^2 - 6y + 9) \\ &= -\pi(y^2 + 2y - 3) \\ &= -\pi(y+3)(y-1), \end{aligned}$$

so the critical point in the interval $(-3, 3)$ is $y = 1$. Since $\frac{dV}{dy} > 0$ for $-3 < y < 1$ and $\frac{dV}{dy} < 0$ for $1 < y < 3$, the critical point does correspond to the maximum value, which is $V(1) = \frac{32\pi}{3}$ cubic units.

37. **(a)** Note that $w^2 + d^2 = 12^2$, so $d = \sqrt{144 - w^2}$. Then we may write $S = kwd^2 = kw(144 - w^2) = 144kw - kw^3$ for $0 < w < 12$, so $\frac{dS}{dw} = 144k - 3kw^2 = -3k(w^2 - 48)$. The critical point (for $0 < w < 12$) occurs at $w = \sqrt{48} = 4\sqrt{3}$. Since $\frac{dS}{dw} > 0$ for $0 < w < 4\sqrt{3}$ and $\frac{dS}{dw} < 0$ for $4\sqrt{3} < w < 12$, the critical point corresponds to the maximum strength. The dimensions are $4\sqrt{3}$ in. wide by $4\sqrt{6}$ in. deep.

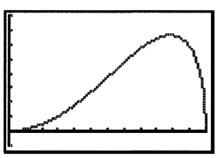
(b)



[0, 12] by [−100, 800]

The graph of $S = 144w - w^3$ is shown. The maximum strength shown in the graph occurs at $w = 4\sqrt{3} \approx 6.9$, which agrees with the answer to part (a).

(c)



[0, 12] by [−100, 800]

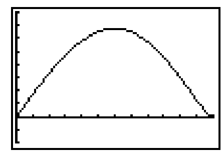
The graph of $S = d^2\sqrt{144 - d^2}$ is shown. The maximum strength shown in the graph occurs at $d = 4\sqrt{6} \approx 9.8$, which agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.
Changing the value of k changes the maximum strength, but not the dimensions of the strongest beam. The graphs for different values of k look the same except that the vertical scale is different.

38. **(a)** Note that $w^2 + d^2 = 12^2$, so $d = \sqrt{144 - w^2}$. Then we may write $S = kwd^3 = kw(144 - w^2)^{3/2}$, so

$$\begin{aligned}\frac{dS}{dw} &= kw \cdot \frac{3}{2}(144 - w^2)^{1/2}(-2w) + k(144 - w^2)^{3/2} \\ &= (k\sqrt{144 - w^2})(-3w^2 + 144 - w^2) \\ &= (-4k\sqrt{144 - w^2})(w^2 - 36)\end{aligned}$$

The critical point (for $0 < w < 12$) occurs at $w = 6$. Since $\frac{dS}{dw} > 0$ for $0 < w < 6$ and $\frac{dS}{dw} < 0$ for $6 < w < 12$, the critical point corresponds to the maximum stiffness. The dimensions are 6 in. wide by $6\sqrt{3}$ in. deep.

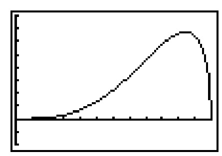
(b)



[0, 12] by [–2000, 8000]

The graph of $S = w(144 - w^2)^{3/2}$ is shown. The maximum stiffness shown in the graph occurs at $w = 6$, which agrees with the answer to part (a).

(c)



[0, 12] by [–2000, 8000]

The graph of $S = d^3\sqrt{144 - d^2}$ is shown. The maximum stiffness shown in the graph occurs at $d = 6\sqrt{3} \approx 10.4$ agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.
Changing the value of k changes the maximum stiffness, but not the dimensions of the stiffest beam. The graphs for different values of k look the same except that the vertical scale is different.

39. **(a)** $v(t) = s'(t) = -10\pi \sin \pi t$

The speed at time t is $10\pi|\sin \pi t|$. The maximum speed is 10π cm/sec and it occurs at $t = \frac{1}{2}$, $t = \frac{3}{2}$, $t = \frac{5}{2}$, and $t = \frac{7}{2}$ sec. The position at these times is $s = 0$ cm (rest position), and the

acceleration $a(t) = v'(t) = -10\pi^2 \cos \pi t$ is 0 cm/sec^2 at these times.

(b) Since $a(t) = -10\pi^2 \cos \pi t$, the greatest magnitude of the acceleration occurs at $t = 0$, $t = 1$, $t = 2$, $t = 3$, and $t = 4$. At these times, the position of the cart is either $s = -10 \text{ cm}$ or $s = 10 \text{ cm}$, and the speed of the cart is 0 cm/sec.

40. Since $\frac{di}{dt} = -2\sin t + 2\cos t$, the largest magnitude of the current occurs when $-2 \sin t + 2 \cos t = 0$, or $\sin t = \cos t$. Squaring both sides gives $\sin^2 t = \cos^2 t$, and we know that $\sin^2 t + \cos^2 t = 1$, so $\sin^2 t = \cos^2 t = \frac{1}{2}$. Thus the possible values of t are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and so on. Eliminating extraneous solutions, the solutions of $\sin t = \cos t$ are $t = \frac{\pi}{4} + k\pi$ for integers k, and at these times $|i| = |2\cos t + 2\sin t| = 2\sqrt{2}$. The peak current is $2\sqrt{2}$ amps.

41. The square of the distance is

$$D(x) = \left(x - \frac{3}{2}\right)^2 + \left(\sqrt{x} - 0\right)^2 = x^2 - 2x + \frac{9}{4},$$

so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

42. Calculus method:
The square of the distance from the point $\left(1, \sqrt{3}\right)$ to $\left(x, \sqrt{16 - x^2}\right)$ is given by

$$\begin{aligned} D(x) &= (x-1)^2 + (\sqrt{16 - x^2} - \sqrt{3})^2 \\ &= x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 \\ &= -2x + 20 - 2\sqrt{48 - 3x^2}. \text{ Then} \end{aligned}$$

$$\begin{aligned} D'(x) &= -2 - \frac{2}{2\sqrt{48 - 3x^2}}(-6x) \\ &= -2 + \frac{6x}{\sqrt{48 - 3x^2}}. \end{aligned}$$

Solving $D'(x) = 0$, we have:

$$\begin{aligned} 6x &= 2\sqrt{48 - 3x^2} \\ 36x^2 &= 4(48 - 3x^2) \\ 9x^2 &= 48 - 3x^2 \\ 12x^2 &= 48 \\ x &= \pm 2 \end{aligned}$$

We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.
Geometry method:
The semicircle is centered at the origin and has radius 4.
The distance from the origin to $\left(1, \sqrt{3}\right)$ is $\sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $\left(1, \sqrt{3}\right)$. That distance is $4 - 2 = 2$.

43. No. Since $f(x)$ is a quadratic function and the coefficient of x^2 is positive, it has an absolute minimum at the point where $f'(x) = 2x - 1 = 0$, and the point is $\left(\frac{1}{2}, \frac{3}{4}\right)$.

44. **(a)** Because $f(x)$ is periodic with period 2π.

(b) No; since $f(x)$ is continuous on $[0, 2\pi]$, its absolute minimum occurs at a critical point or endpoint.
Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x) = -4\sin x - 2\sin 2x &= 0 \\ -4\sin x - 4\sin x \cos x &= 0 \\ -4(\sin x)(1 + \cos x) &= 0 \\ \sin x &= 0 \text{ or } \cos x = -1 \\ x &= 0, \pi, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 8)$, $(\pi, 0)$, and $(2\pi, 8)$. Thus, $f(x)$ has an absolute minimum at $(\pi, 0)$ and it is never negative.

45. **(a)**

$$\begin{aligned} 2\sin t &= \sin 2t \\ 2\sin t &= 2\sin t \cos t \\ 2(\sin t)(1 - \cos t) &= 0 \\ \sin t &= 0 \text{ or } \cos t = 1 \end{aligned}$$

$t = k\pi$, where k is an integer.
The masses pass each other whenever t is an integer multiple of π seconds.

(b) The vertical distance between the objects is the absolute value of $f(x) = \sin 2t - 2\sin t$.

Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x) = 2\cos 2t - 2\cos t &= 0 \\ 2(2\cos^2 t - 1) - 2\cos t &= 0 \\ 2(2\cos^2 t - \cos t - 1) &= 0 \\ 2(2\cos t + 1)(\cos t - 1) &= 0 \\ \cos t &= -\frac{1}{2} \text{ or } \cos t = 1 \\ t &= \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 0)$, $\left(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{2}\right)$, $\left(\frac{4\pi}{3}, \frac{3\sqrt{3}}{2}\right)$, and $(2\pi, 0)$

The distance is greatest when $t = \frac{2\pi}{3}$ sec and when $t = \frac{4\pi}{3}$ sec. The distance at those times is $\frac{3\sqrt{3}}{2}$ meters.

46. (a)

$$\begin{aligned} \sin t &= \sin\left(t + \frac{\pi}{3}\right) \\ \sin t &= \sin t \cos\frac{\pi}{3} + \cos t \sin\frac{\pi}{3} \\ \sin t &= \frac{1}{2}\sin t + \frac{\sqrt{3}}{2}\cos t \\ \frac{1}{2}\sin t &= \frac{\sqrt{3}}{2}\cos t \\ \tan t &= \sqrt{3} \end{aligned}$$

Solving for *t*, the particles meet at $t = \frac{\pi}{3}$ sec and at $t = \frac{4\pi}{3}$ sec.

(b) The distance between the particles is the absolute value of $f(t) = \sin\left(t + \frac{\pi}{3}\right) - \sin t = \frac{\sqrt{3}}{2}\cos t - \frac{1}{2}\sin t$.

Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(t) = -\frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t &= 0 \\ -\frac{\sqrt{3}}{2}\sin t &= \frac{1}{2}\cos t \\ \tan t &= -\frac{1}{\sqrt{3}} \end{aligned}$$

The solutions are $t = \frac{5\pi}{6}$ and $t = \frac{11\pi}{6}$, so the critical points are at $\left(\frac{5\pi}{6}, -1\right)$ and $\left(\frac{11\pi}{6}, 1\right)$, and the interval endpoints are at $\left(0, \frac{\sqrt{3}}{2}\right)$, and $\left(2\pi, \frac{\sqrt{3}}{2}\right)$. The particles are farthest apart at $t = \frac{5\pi}{6}$ sec and at $t = \frac{11\pi}{6}$ sec, and the maximum distance between the particles is 1 m.

(c) We need to maximize $f'(t)$, so we solve $f''(t) = 0$.

$$\begin{aligned} f''(t) = -\frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t &= 0 \\ \frac{1}{2}\sin t &= \frac{\sqrt{3}}{2}\cos t \end{aligned}$$

This is the same equation we solved in part (a), so the solutions are $t = \frac{\pi}{3}$ sec and $t = \frac{4\pi}{3}$ sec.

For the function $y = f'(t)$, the critical points occur at $\left(\frac{\pi}{3}, -1\right)$ and $\left(\frac{4\pi}{3}, 1\right)$, and the interval endpoints are at $\left(0, -\frac{1}{2}\right)$ and $\left(2\pi, -\frac{1}{2}\right)$.

Thus, $|f'(t)|$ is maximized at $t = \frac{\pi}{3}$ and $t = \frac{4\pi}{3}$. But these are the instants when the particles pass each other, so the graph of $y = |f(t)|$ has corners at these points and $\frac{d}{dt}|f(t)|$ is undefined at these instants. We cannot say that the distance is changing the fastest at any particular instant, but we can say that

near $t=\frac{\pi}{3}$ or $t=\frac{4\pi}{3}$ the distance is changing faster than at any other time in the interval.

47. The trapezoid has height $(\cos\theta)$ft and the trapezoid bases measure 1 ft and $(1+2\sin\theta)$ft, so the volume is given by

$$V(\theta)=\frac{1}{2}(\cos\theta)(1+1+2\sin\theta)(20)$$
$$=20(\cos\theta)(1+\sin\theta).$$

Find the critical points for $0\le\theta<\frac{\pi}{2}$:

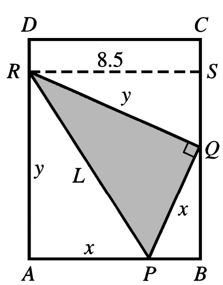
$$V'(\theta)$$
$$=20(\cos\theta)(\cos\theta)+20(1+\sin\theta)(-\sin\theta)$$
$$=0$$

$$20\cos^2\theta-20\sin\theta-20\sin^2\theta=0$$
$$20(1-\sin^2\theta)-20\sin\theta-20\sin^2\theta=0$$
$$-20(2\sin^2\theta+\sin\theta-1=0$$
$$-20(2\sin\theta-1)(\sin\theta+1)=0$$

$$\sin\theta=\frac{1}{2}\text{ or }\sin\theta=-1$$
$$\theta=\frac{\pi}{6}$$

The critical point is at $\left(\frac{\pi}{6},15\sqrt{3}\right)$. Since $V'(\theta)>0$ for $0\le\theta<\frac{\pi}{6}$ and $V'(\theta)<0$ for $\frac{\pi}{6}<\theta<\frac{\pi}{2}$, the critical point corresponds to the maximum possible trough volume. The volume is maximized when $\theta=\frac{\pi}{6}$.

48.



Sketch segment RS as shown, and let y be the length of segment QR. Note that $PB=8.5-x$, and so

$$QB=\sqrt{x^2-(8.5-x)^2}=\sqrt{8.5(2x-8.5)}.$$

Also note that triangles QRS and PQB are similar.

$$\frac{QR}{RS}=\frac{PQ}{QB}$$
$$\frac{y}{8.5}=\frac{x}{\sqrt{8.5(2x-8.5)}}$$

(a)
$$\frac{y^2}{8.5^2}=\frac{x^2}{8.5(2x-8.5)}$$
$$y^2=\frac{8.5x^2}{2x-8.5}$$
$$L^2=x^2+y^2$$
$$L^2=x^2+\frac{8.5x^2}{2x-8.5}$$
$$L^2=\frac{x^2(2x-8.5)+8.5x^2}{2x-8.5}$$
$$L^2=\frac{2x^3}{2x-8.5}$$

(b) Note that $x>4.25$, and let $f(x)=L^2=\frac{2x^3}{2x-8.5}$. Since $y\le 11$, the approximate domain of f is $5.20\le x\le 8.5$. Then

$$f'(x)=\frac{(2x-8.5)(6x^2)-(2x^3)(2)}{(2x-8.5)^2}$$
$$=\frac{x^2(8x-51)}{(2x-8.5)^2}$$

For $x>5.20$, the critical point occurs at $x=\frac{51}{8}=6.375$ in., and this corresponds to a minimum value of $f(x)$ because $f'(x)<0$ for $5.20<x<6.375$ and $f'(x)>0$ for $x>6.375$. Therefore, the value of x that minimizes L^2 is $x=6.375$ in.

(c) The minimum value of L is
$$\sqrt{\frac{2(6.375)^3}{2(6.375)-8.5}}\approx 11.04\text{ in.}$$

49. Since $R=M^2\left(\frac{C}{2}-\frac{M}{3}\right)=\frac{C}{2}M^2-\frac{1}{3}M^3$, we have $\frac{dR}{dM}=CM-M^2$. Let $f(M)=CM-M^2$. Then $f'(M)=C-2M$,

and the critical point for f occurs at $M = \frac{C}{2}$.
This value corresponds to a maximum because $f'(M) > 0$ for $M < \frac{C}{2}$ and $f'(M) < 0$ for $M > \frac{C}{2}$.
The value of M that maximizes $\frac{dR}{dM}$ is $M = \frac{C}{2}$.

50. The profit is given by

$$\begin{aligned} P(x) &= (n)(x-c) \\ &= a + b(100-x)(x-c) \\ &= -bx^2 + (100+c)bx + (a - 100bc). \end{aligned}$$

Then $P'(x) = -2bx + (100+c)b$
$= b(100 + c - 2x).$

The critical point occurs at $x = \frac{100+c}{2} = 50 + \frac{c}{2}$, and this value corresponds to the maximum profit because $P'(x) > 0$ for $x < 50 + \frac{c}{2}$ and $P'(x) < 0$ for $x > 50 + \frac{c}{2}$.

A selling price of $50 + \frac{c}{2}$ will bring the maximum profit.

51. True. This is guaranteed by the Extreme Value Theorem (Section 5.1).

52. False; for example, consider $f(x) = x^3$ at $c = 0$.

53. D; $f(x) = x^2(60-x)$

$$\begin{aligned} f'(x) &= x^2(-1) + (60-x)(2x) \\ &= -x^2 + 120x - 2x^2 \\ &= -3x^2 + 120x \\ &= -3x(x-40) \end{aligned}$$

$x = 0$ or $x = 40$
$60 - x = 60$ $\quad 60 - x = 20$
$x^2(60-x) = 0$
$(40)^2(20) = (1600)(20)$
$= 32{,}000$

54. B; since $f'(x)$ is negative, $f(x)$ is always decreasing, so $f(25) = 3$.

55. B; $A = \frac{1}{2}bh$

$$\begin{aligned} b^2 + h^2 &= 100 \\ b &= \sqrt{100-h^2} \\ A &= \frac{h}{2}\sqrt{100-h^2} \\ A' &= \frac{\sqrt{100-h^2}}{2} - \frac{h^2}{2\sqrt{100-h^2}} \end{aligned}$$

$A' = 0$ when $h = \sqrt{50}$

$b = \sqrt{100 - \sqrt{50}^2} = \sqrt{50}$

$A_{\max} = \frac{1}{2}\sqrt{50}\sqrt{50} = 25$

56. E; length $= 2x$

height $= 30 - x^2 - 4x^2 = 30 - 5x^2$

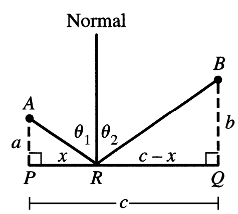
area $= A = 2x(30 - 5x^2) = 60x - 10x^3$

$\frac{dA}{dx}(60x - 10x^3) = 60 - 30x^2$

$x = \sqrt{2}$

$2\sqrt{2}\left(30 - 5(\sqrt{2})^2\right) = 40\sqrt{2}.$

57.



Let P be the foot of the perpendicular from A to the mirror, and Q be the foot of the perpendicular from B to the mirror. Suppose the light strikes the mirror at point R on the way from A to B. Let:
a = distance from A to P
b = distance from B to Q
c = distance from P to Q
x = distance from P to R
To minimize the time is to minimize the total distance the light travels going from A to B. The total distance is

$$D(x) = \sqrt{x^2 + a^2} + \sqrt{(c-x)^2 + b^2}$$

Then

$$D'(x) = \frac{1}{2\sqrt{x^2+a^2}}(2x) + \frac{1}{2\sqrt{(c-x)^2+b^2}}[-2(c-x)]$$
$$= \frac{x}{\sqrt{x^2+a^2}} - \frac{c-x}{\sqrt{(c-x)^2+b^2}}$$

Solving $D'(x) = 0$ gives the equation $\frac{x}{\sqrt{x^2+a^2}} = \frac{c-x}{\sqrt{(c-x)^2+b^2}}$ which we will refer to as Equation 1.

Squaring both sides, we have:

$$\frac{x^2}{x^2+a^2} = \frac{(c-x)^2}{(c-x)^2+b^2}$$
$$x^2\left[(c-x)^2+b^2\right] = (c-x)^2(x^2+a^2)$$
$$x^2(c-x)^2 + x^2b^2 = (c-x)^2x^2 + (c-x)^2a^2$$
$$x^2b^2 = (c-x)^2a^2$$
$$x^2b^2 = \left[c^2-2xc+x^2\right]a^2$$
$$0 = (a^2-b^2)x^2 - 2a^2cx + a^2c^2$$
$$0 = [(a+b)x-ac][(a-b)x-ac]$$
$$x = \frac{ac}{a+b} \text{ or } x = \frac{ac}{a-b}$$

Note that the value $x = \frac{ac}{a-b}$ is an extraneous solution because $c - x = c - \frac{ac}{a-b} = \frac{-cb}{a-b}$, so x and $c - x$ could not both be positive. The only critical point occurs at $x = \frac{ac}{a+b}$.

To verify that critical point represents the minimum distance, notice that D is differentiable for all x in $[0, c]$ with a single critical point in the interior of the interval. Since $D'(0) = \frac{-c}{\sqrt{c^2+b^2}} < 0$, D must be decreasing from 0 to the critical point, and since $D'(c) = \frac{c}{\sqrt{c^2+b^2}} > 0$, D must be increasing from the critical point to c.

We now know that $D(x)$ is minimized when Equation 1 is true, or, equivalently, $\frac{PR}{AR} = \frac{QR}{BR}$. This means that the two right triangles APR and BQR are similar, which in turn implies that the two angles must be equal.

58. $\frac{dv}{dx} = ka - 2kx$

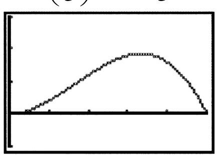
The critical point occurs at $x = \frac{ka}{2k} = \frac{a}{2}$, which represents a maximum value because $\frac{d^2v}{dx^2} = -2k$, which is negative for all x. The maximum value of v is $kax - kx^2 = ka\left(\frac{a}{2}\right) - k\left(\frac{a}{2}\right)^2 = \frac{ka^2}{4}$.

59. **(a)** $v = cr_0r^2 - cr^3$

$$\frac{dv}{dr} = 2cr_0r - 3cr^2 = cr(2r_0 - 3r)$$

The critical point occurs at $r = \frac{2r_0}{3}$. (Note that $r = 0$ is not in the domain of v.) The critical point represents a maximum because $\frac{d^2v}{dr^2} = 2cr_0 - 6cr = 2c(r_0 - 3r)$, which is negative when $r = \frac{2r_0}{3}$.

(b) We graph $v = (0.5 - r)r^2$, and observe that the maximum indeed occurs at $v = \left(\frac{2}{3}\right)0.5 = \frac{1}{3}$.



[0, 0.5] by [−0.01, 0.03]

60. **(a)** Since $A'(q) = -kmq^{-2} + \frac{h}{2}$, the critical point occurs when $\frac{km}{q^2} = \frac{h}{2}$, or $q = \sqrt{\frac{2km}{h}}$. This corresponds to the minimum value of $A(q)$ because $A''(q) = 2kmq^{-3}$, which is positive for $q > 0$.

(b) The new formula for average weekly cost is $B(q) = \frac{(k + bq)m}{q} + cm + \frac{hq}{2}$

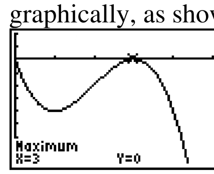
$$= \frac{km}{q} + bm + cm + \frac{hq}{2}$$
$$= A(q) + bm$$

Since $B(q)$ differs from $A(q)$ by a constant, the minimum value of $B(q)$ will occur at the same q-value as the minimum value of $A(q)$. The most economical quantity is again $\sqrt{\frac{2km}{h}}$.

61. The profit is given by

$$p(x) = r(x) - c(x)$$
$$= 6x - (x^3 - 6x^2 + 15x)$$
$$= -x^3 + 6x^2 - 9x, \text{ for } x \geq 0.$$

Then $p'(x) = -3x^2 + 12x - 9 = -3(x-1)(x-3)$, so the critical points occur at $x = 1$ and $x = 3$. Since $p'(x) < 0$ for $0 \leq x < 1$, $p'(x) > 0$ for $1 < x < 3$, and $p'(x) < 0$ for $x > 3$, the relative maxima occur at the endpoint $x = 0$ and at the critical point $x = 3$. Since $p(0) = p(3) = 0$, this means that for $x \geq 0$, the function $p(x)$ has its absolute maximum value at the points (0, 0) and (3, 0). This result can also be obtained graphically, as shown.



[0, 5] by [−8, 2]

62. The average cost is given by $a(x) = \frac{c(x)}{x} = x^2 - 20x + 20{,}000$. Therefore, $a'(x) = 2x - 20$ and the critical value is $x = 10$, which represents the minimum because $a''(x) = 2$, which is positive for all x. The average cost is minimized at a production level of 10 items.

63. **(a)** According to the graph, $y'(0) = 0$.

(b) According to the graph, $y'(-L) = 0$.

(c) $y(0) = 0$, so $d = 0$.

Now $y'(x) = 3ax^2 + 2bx + c$, so $y'(0) = 0$ implies that $c = 0$. Therefore, $y(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$. Then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 - 2bL = 0$, so we have two linear equations in the two unknowns a and b. The second equation gives $b = \frac{3aL}{2}$. Substituting into the first equation, we have $-aL^3 + \frac{3aL^3}{2} = H$, or $\frac{aL^3}{2} = H$, so $a = 2\frac{H}{L^3}$. Therefore,

$b = 3\dfrac{H}{L^2}$ and the equation for y is $y(x) = 2\dfrac{H}{L^3}x^3 + 3\dfrac{H}{L^2}x^2$, or $y(x) = H\left[2\left(\dfrac{x}{L}\right)^3 + 3\left(\dfrac{x}{L}\right)^2\right]$.

64. (a) The base radius of the cone is $r = \dfrac{2\pi a - x}{2\pi}$ and so the height is $h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \left(\dfrac{2\pi a - x}{2\pi}\right)^2}$.

Therefore, $V(x) = \dfrac{\pi}{3}r^2 h = \dfrac{\pi}{3}\left(\dfrac{2\pi a - x}{2\pi}\right)^2 \sqrt{a^2 - \left(\dfrac{2\pi a - x}{2\pi}\right)^2}$.

(b) To simplify the calculations, we shall consider the volume as a function of *r:*

volume $= f(r) = \dfrac{\pi}{3}r^2\sqrt{a^2 - r^2}$, where $0 < r < a$.

$$\begin{aligned} f'(r) &= \frac{\pi}{3}\frac{d}{dr}\left(r^2\sqrt{a^2 - r^2}\right) \\ &= \frac{\pi}{3}\left[r^2 \cdot \frac{1}{2\sqrt{a^2 - r^2}} \cdot (-2r) + \left(\sqrt{a^2 - r^2}\right)(2r)\right] \\ &= \frac{\pi}{3}\left[\frac{-r^3 + 2r(a^2 - r^2)}{\sqrt{a^2 - r^2}}\right] \\ &= \frac{\pi}{3}\left[\frac{(2a^2 r - 3r^3)}{\sqrt{a^2 - r^2}}\right] \\ &= \frac{\pi r(2a^2 - 3r^2)}{3\sqrt{a^2 - r^2}} \end{aligned}$$

The critical point occurs when $r^2 = \dfrac{2a^2}{3}$, which gives $r = a\sqrt{\dfrac{2}{3}} = \dfrac{a\sqrt{6}}{3}$. Then

$h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \dfrac{2a^2}{3}} = \sqrt{\dfrac{a^2}{3}} = \dfrac{a\sqrt{3}}{3}$. Using $r = \dfrac{a\sqrt{6}}{3}$ and $h = \dfrac{a\sqrt{3}}{3}$, we may now find the values of r and h for the given values of a

when $a = 4: r = \dfrac{4\sqrt{6}}{3}, h = \dfrac{4\sqrt{3}}{3}$; when $a = 5: r = \dfrac{5\sqrt{6}}{3}, h = \dfrac{5\sqrt{3}}{3}$;

when $a = 6: r = 2\sqrt{6}, h = 2\sqrt{3}$; when $a = 8: r = \dfrac{8\sqrt{6}}{3}, h = \dfrac{8\sqrt{3}}{3}$

(c) Since $r = \dfrac{a\sqrt{6}}{3}$ and $h = \dfrac{a\sqrt{3}}{3}$, the relationship is $\dfrac{r}{h} = \sqrt{2}$.

65. (a) Let x_0 represent the fixed value of x at point P, so that P has coordinates (x_0, a) and let $m = f'(x_0)$ be the slope of line RT. Then the equation of line RT is $y = m(x - x_0) + a$. The y-intercept of this line is $m(0 - x_0) + a = a - mx_0$, and the x-intercept is the solution of $m(x - x_0) + a = 0$, or $x = \dfrac{mx_0 - a}{m}$. Let O designate the origin. Then

(Area of triangle RST) = 2 (Area of triangle ORT)

$$= 2 \cdot \frac{1}{2} \text{ (x-intercept of line RT) (y-intercept of line RT)}$$

$$= 2 \cdot \frac{1}{2}\left(\frac{mx_0 - a}{m}\right)(a - mx_0)$$

$$= -m\left(\frac{mx_0 - a}{m}\right)\left(\frac{mx_0 - a}{m}\right)$$

$$= -m\left(\frac{mx_0 - a}{m}\right)^2$$

$$= -m\left(x_0 - \frac{a}{m}\right)^2$$

Substituting x for x_0, $f'(x)$ for m, and $f(x)$ for a, we have $A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$.

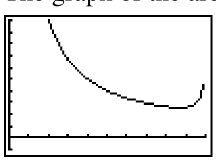
(b) The domain is the open interval (0, 10).

To graph, let $y_1 = f(x) = 5 + 5\sqrt{1 - \frac{x^2}{100}}$,

$y_2 = f'(x) = \text{NDER}(y_1)$, and

$y_3 = A(x) = -y_2\left(x - \frac{y_1}{y^2}\right)^2$.

The graph of the area function $y_3 = A(x)$ is shown below.



[0, 10] by [–100, 1000]

The vertical asymptotes at $x = 0$ and $x = 10$ correspond to horizontal or vertical tangent lines, which do not form triangles.

(c) Using our expression for the y-intercept of the tangent line, the height of the triangle is

$$a - mx = f(x) - f'(x) \cdot x$$

$$= 5 + \frac{1}{2}\sqrt{100 - x^2} - \frac{-x}{2\sqrt{100 - x^2}}x$$

$$= 5 + \frac{1}{2}\sqrt{100 - x^2} + \frac{x^2}{2\sqrt{100 - x^2}}$$

We may use graphing methods or the analytic method in part (d) to find that the minimum value of $A(x)$ occurs at $x \approx 8.66$. Substituting this value into the expression above, the height of the triangle is 15. This is 3 times the y-coordinate of the center of the ellipse.

(d) Part (a) remains unchanged. The domain is (0, C). To graph, note that

$$f(x) = B + B\sqrt{1 - \frac{x^2}{C^2}} = B + \frac{B}{C}\sqrt{C^2 - x^2} \text{ and } f'(x) = \frac{B}{C}\frac{1}{2\sqrt{C^2 - x^2}}(-2x) = \frac{-Bx}{C\sqrt{C^2 - x^2}}.$$

Therefore, we have

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{B + \frac{B}{C}\sqrt{C^2 - x^2}}{\frac{-Bx}{C\sqrt{C^2 - x^2}}}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{(BC + B\sqrt{C^2 - x^2})\sqrt{C^2 - x^2}}{-Bx}\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + (BC + B\sqrt{C^2 - x^2})(\sqrt{C^2 - x^2})\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + BC\sqrt{C^2 - x^2} + B(C^2 - x^2)\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[BC(C + \sqrt{C^2 - x^2})\right]^2$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})^2}{x\sqrt{C^2 - x^2}}$$

$$A'(x) = \frac{BC\left(x\sqrt{C^2 - x^2}\right)(2)\left(C + \sqrt{C^2 - x^2}\right)\left(\frac{-x}{\sqrt{C^2 - x^2}}\right) - BC \cdot \left(C + \sqrt{C^2 - x^2}\right)^2\left(x\frac{-x}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}(1)\right)}{x^2(C^2 - x^2)}$$

$$= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)}{x^2(C^2 - x^2)}\left[-2x^2 - \left(C + \sqrt{C^2 - x^2}\right)\left(\frac{-x^2}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}\right)\right]$$

$$= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)}{x^2\sqrt{C^2 - x^2}}\left[-2x^2 + \frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} + x^2 - (C^2 - x^2)\right]$$

$$= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)}{x^2(C^2 - x^2)}\left(\frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} - C^2\right)$$

$$= \frac{BC\left(C + \sqrt{C^2 - x^2}\right)}{x^2(C^2 - x^2)^{3/2}}\left[Cx^2 - C(C^2 - x^2) - C^2\sqrt{C^2 - x^2}\right]$$

$$= \frac{BC^2\left(C + \sqrt{C^2 - x^2}\right)}{x^2(C^2 - x^2)^{3/2}}\left(2x^2 - C^2 - C\sqrt{C^2 - x^2}\right)$$

To find the critical points for $0 < x < C$, we solve:

$$\begin{aligned} 2x^2 - C^2 &= C\sqrt{C^2 - x^2} \\ 4x^4 - 4C^2x^2 + C^4 &= C^4 - C^2x^2 \\ 4x^4 - 3C^2x^2 &= 0 \\ x^2(4x^2 - 3C^2) &= 0 \end{aligned}$$

The minimum value of $A(x)$ for $0 < x < C$ occurs at the critical point $x = \frac{C\sqrt{3}}{2}$, or $x^2 = \frac{3C^2}{4}$. The corresponding triangle height is

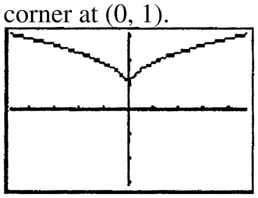
$$
\begin{aligned}
&a-mx\\
&=f(x)-f'(x)\cdot x\\
&=B+\frac{B}{C}\sqrt{C^2-x^2}+\frac{Bx^2}{C\sqrt{C^2-x^2}}\\
&=B+\frac{B}{C}\sqrt{C^2-\frac{3C^2}{4}}+\frac{B\left(\frac{3C^2}{4}\right)}{C\sqrt{C^2-\frac{3C^2}{4}}}\\
&=B+\frac{B}{C}\left(\frac{C}{2}\right)+\frac{\frac{3BC^2}{4}}{\frac{C^2}{2}}\\
&=B+\frac{B}{2}+\frac{3B}{2}\\
&=3B
\end{aligned}
$$

This shows that the triangle has minimum area when its height is $3B$.

Section 5.5 Linearization and Differentials (pp. 237–249)

Exploration 1 Appreciating Local Linearity

1. The graph appears to have either a cusp or a corner at (0, 1).



$y=\left(x^2+0.0001\right)^{1/4}+0.9$

2. $f'(x)=\frac{1}{4}(x^2+0.0001)^{-3/4}(2x)$

$$=\frac{x}{\sqrt[4]{\left(x^2+0.0001\right)^3}}.$$

Since $f'(0)=0,$ the tangent line at (0, 1) has equation $y = 1$.

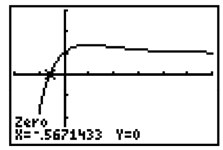
3. The "corner" becomes smooth and the graph straightens out.

4. As with any differentiable curve, the graph comes to resemble the tangent line.

Quick Review 5.5

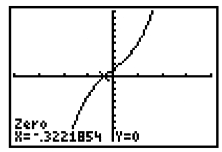
1. $\frac{dy}{dx}=\cos(x^2+1)\cdot\frac{d}{dx}(x^2+1)$
$=2x\cos(x^2+1)$

2. $$\begin{aligned}\frac{dy}{dx}&=\frac{(x+1)(1-\sin x)-(x+\cos x)(1)}{(x+1)^2}\\&=\frac{x-x\sin x+1-\sin x-x-\cos x}{(x+1)^2}\\&=\frac{1-\cos x-(x+1)\sin x}{(x+1)^2}\end{aligned}$$

3. 

[−2, 6] by [−3, 3]

$x\approx-0.567$

4. 

[−4, 4] by [−10, 10]

$x\approx-0.322$

5. $f'(x)=(x)(-e^{-x})+(e^{-x})(1)=e^{-x}-xe^{-x}$

$f'(0)=1$

The lines passes through (0, 1) and has slope 1. Its equation is $y = x + 1$.

6. $f'(x)=(x)(-e^{-x})+(e^{-x})(1)=e^{-x}-xe^{-x}$

$f'(-1)=e^1-(-e^1)=2e$

The lines passes through $(-1, -e + 1)$ and has slope $2e$.

Its equation is $y = 2e(x + 1) + (-e + 1)$, or $y = 2ex + e + 1$.

7. **(a)** $x+1=0$
$x=-1$

(b) $$\begin{aligned}2ex+e+1&=0\\2ex&=-(e+1)\\x&=-\frac{e+1}{2e}\\&\approx-0.684\end{aligned}$$

8. $f'(x)=3x^2-4$

$f'(1)=3(1)^2-4=-1$

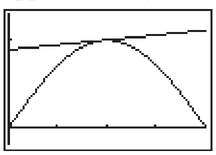
Since $f(1)=-2$ and $f'(1)=-1,$ the graph of $g(x)$ passes through $(1, -2)$ and has slope -1. Its equation is $g(x) = -1(x - 1) + (-2)$, or $g(x) = -x - 1$.

x	$f(x)$	$g(x)$
0.7	−1.457	−1.7
0.8	−1.688	−1.8
0.9	−1.871	−1.9
1.0	−2	−2
1.1	−2.069	−2.1
1.2	−2.072	−2.2
1.3	−2.003	−2.3

9. $f'(x) = \cos x$

$f'(1.5) = \cos 1.5$

Since $f(1.5) = \sin 1.5$ and $f'(1.5) = \cos 1.5$, the tangent line passes through (1.5, sin 1.5) and has slope cos 1.5. Its equation is $y = (\cos 1.5)(x - 1.5) + \sin 1.5$, or approximately $y = 0.071x + 0.891$

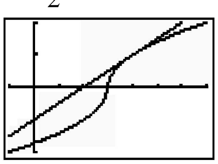


[0, π] by [–0.2, 1.3]

10. For $x > 3$, $f'(x) = \dfrac{1}{2\sqrt{x-3}}$, and so

$f'(4) = \dfrac{1}{2}$. Since $f(4) = 1$ and $f'(4) = \dfrac{1}{2}$, the tangent line passes through (4, 1) and has slope $\dfrac{1}{2}$. Its equation is $y = \dfrac{1}{2}(x-4) + 1$, or

$y = \dfrac{1}{2}x - 1.$



[−1, 7] by [−2, 2]

Section 5.5 Exercises

1. (a) $f'(x) = 3x^2 - 2$

We have $f(2) = 7$ and $f'(2) = 10$.

$$\begin{aligned} L(x) &= f(2) + f'(2)(x-2) \\ &= 7 + 10(x-2) \\ &= 10x - 13 \end{aligned}$$

(b) Since $f(2.1) = 8.061$ and $L(2.1) = 8$, the approximation differs from the true value in absolute value by less than 10^{-1}.

2. (a) $f'(x) = \dfrac{1}{2\sqrt{x^2+9}}(2x) = \dfrac{x}{\sqrt{x^2+9}}$

We have $f(-4) = 5$ and $f'(-4) = -\dfrac{4}{5}$.

$$\begin{aligned} L(x) &= f(-4) + f'(-4)(x-(-4)) \\ &= 5 - \frac{4}{5}(x+4) \\ &= -\frac{4}{5}x + \frac{9}{5} \end{aligned}$$

(b) Since $f(-3.9) \approx 4.9204$ and $L(-3.9) = 4.92$, the approximation differs from the true value by less than 10^{-3}.

3. (a) $f'(x) = 1 - x^{-2}$

We have $f(1) = 2$ and $f'(1) = 0$.

$$\begin{aligned} L(x) &= f(1) + f'(1)(x-1) \\ &= 2 + 0(x-1) \\ &= 2 \end{aligned}$$

(b) Since $f(1.1) = 2.009$ and $L(1.1) = 2$, the approximation differs from the true value by less than 10^{-2}.

4. (a) $f'(x) = \dfrac{1}{x+1}$

We have $f(0) = 0$ and $f'(0) = 1$.

$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= 0 + 1x \\ &= x \end{aligned}$$

(b) Since $f(0.1) \approx 0.0953$ and $L(0.1) = 0.1$ the approximation differs from the true value by less than 10^{-2}.

5. **(a)** $f'(x) = \sec^2 x$

We have $f(\pi) = 0$ and $f'(\pi) = 1$.

$$\begin{aligned} L(x) &= f(\pi) + f'(\pi)(x-\pi) \\ &= 0 + 1(x-\pi) \\ &= x - \pi \end{aligned}$$

(b) Since $f(\pi + 0.1) \approx 0.10033$ and $L(\pi + 0.1) = 0.1$, the approximation differs from the true value in absolute value by less than 10^{-3}.

6. **(a)** $f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$

We have $f(0) = \dfrac{\pi}{2}$ and $f'(0) = -1$.

$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= \frac{\pi}{2} + (-1)(x-0) \\ &= -x + \frac{\pi}{2} \end{aligned}$$

(b) Since $f(0.1) \approx 1.47063$ and $L(0.1) \approx 1.47080$, the approximation differs from the true value in absolute value by less than 10^{-3}.

7. $f'(x) = k(1+x)^{k-1}$

We have $f(0) = 1$ and $f'(0) = k$.

$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= 1 + k(x-0) \\ &= 1 + kx \end{aligned}$$

8. **(a)**
$$\begin{aligned} (1.002)^{100} &= (1+0.002)^{100} \\ &\approx 1 + (100)(0.002) \\ &= 1.2; \end{aligned}$$
$$\left|1.002^{100} - 1.2\right| \approx 0.021 < 10^{-1}$$

(b)
$$\begin{aligned} \sqrt[3]{1.009} &= (1+0.009)^{1/3} \\ &\approx 1 + \frac{1}{3}(0.009) \\ &= 1.003; \end{aligned}$$
$$\left|\sqrt[3]{1.009} - 1.003\right| \approx 9 \times 10^{-6} < 10^{-5}$$

9. **(a)**
$$\begin{aligned} f(x) &= (1-x)^6 \\ &= [1+(-x)]^6 \\ &\approx 1 + 6(-x) \\ &= 1 - 6x \end{aligned}$$

(b)
$$\begin{aligned} f(x) &= \frac{2}{1-x} \\ &= 2[1+(-x)]^{-1} \\ &\approx 2[1+(-1)(-x)] \\ &= 2 + 2x \end{aligned}$$

(c) $f(x) = (1+x)^{-1/2} \approx 1 + \left(-\dfrac{1}{2}\right)x = 1 - \dfrac{x}{2}$

10. **(a)**
$$\begin{aligned} f(x) &= (4+3x)^{1/3} \\ &= 4^{1/3}\left(1 + \frac{3x}{4}\right)^{1/3} \\ &\approx 4^{1/3}\left(1 + \frac{1}{3}\left(\frac{3x}{4}\right)\right) \\ &= 4^{1/3}\left(1 + \frac{x}{4}\right) \end{aligned}$$

(b)
$$\begin{aligned} f(x) &= \sqrt{2+x^2} \\ &= \sqrt{2}\left(1 + \frac{x^2}{2}\right)^{1/2} \\ &\approx \sqrt{2}\left(1 + \frac{1}{2}\left(\frac{x^2}{2}\right)\right) \\ &= \sqrt{2}\left(1 + \frac{x^2}{4}\right) \end{aligned}$$

(c)
$$\begin{aligned} f(x) &= \left(1 - \frac{1}{2+x}\right)^{2/3} \\ &= \left[1 + \left(-\frac{1}{2+x}\right)\right]^{2/3} \\ &\approx 1 + \frac{2}{3}\left(-\frac{1}{2+x}\right) \\ &= 1 - \frac{2}{6+3x} \end{aligned}$$

11. $x = 100$

$f'(100) = \dfrac{1}{2}(100)^{-1/2} = 0.05$

$f(100) \approx 10 + 0.05(101 - 100) = 10.05$

12. $x = 27$

$f'(27) = \dfrac{1}{3}(27)^{-2/3} = \dfrac{1}{27}$

$f(27) \approx 3 + \left(\dfrac{1}{27}\right)(26 - 27)$

$y = 3 - \dfrac{1}{27} \approx 2.\overline{962}$

13. $x = 1000$

$$f'(1000) = \frac{1}{3}(1000)^{-2/3} = \frac{1}{300}$$

$$y = 10 + \left(\frac{1}{300}\right)(x - 1000)$$

$$y = 10 - \frac{1}{150} = 9.99\overline{3}$$

14. $x = 81$

$$f'(81) = \frac{1}{2}(81)^{-1/2} = \frac{1}{18}$$

$$y = 9 + \frac{1}{18}(80 - 81)$$

$$y = 9 - \frac{1}{18} = 8.9\overline{4}$$

15. (a) Since $\dfrac{dy}{dx} = 3x^2 - 3$, $dy = (3x^2 - 3)\,dx$.

(b) At the given values,

$$dy = (3 \cdot 2^2 - 3)(0.05) = 9(0.05) = 0.45.$$

16. (a) Since

$$\frac{dy}{dx} = \frac{(1+x^2)(2) - (2x)(2x)}{(1+x^2)^2} = \frac{2 - 2x^2}{(1+x^2)^2},$$

$$dy = \frac{2 - 2x^2}{(1+x^2)^2}\,dx.$$

(b) At the given values,

$$dy = \frac{2 - 2(-2)^2}{[1 + (-2)^2]^2}(0.1)$$

$$= \frac{2 - 8}{5^2}(0.1)$$

$$= -0.024.$$

17. (a) Since

$$\frac{dy}{dx} = (x^2)\left(\frac{1}{x}\right) + (\ln x)(2x) = 2x \ln x + x,$$

$$dy = (2x \ln x + x)dx.$$

(b) At the given values,

$$dy = [2(1)\ln(1) + 1](0.01) = 1(0.01) = 0.01$$

18. (a) Since

$$\frac{dy}{dx} = (x)\left(\frac{1}{2\sqrt{1 - x^2}}\right)(-2x) + (\sqrt{1 - x^2})(1)$$

$$= \frac{-x^2}{\sqrt{1 - x^2}} + \sqrt{1 - x^2}$$

$$= \frac{-x^2 + (1 - x^2)}{\sqrt{1 - x^2}}$$

$$= \frac{1 - 2x^2}{\sqrt{1 - x^2}},$$

$$dy = \frac{1 - 2x^2}{\sqrt{1 - x^2}}\,dx.$$

(b) At the given values,

$$dy = \frac{1 - 2(0)^2}{\sqrt{1 - (0)^2}}(-0.2) = -0.2.$$

19. (a) Since $\dfrac{dy}{dx} = e^{\sin x}\cos x$,

$$dy = (\cos x)e^{\sin x}dx.$$

(b) At the given values,

$$dy = (\cos \pi)(e^{\sin \pi})(-0.1)$$

$$= (-1)(1)(-0.1)$$

$$= 0.1.$$

20. (a) Since

$$\frac{dy}{dx} = -3\csc\left(1 - \frac{x}{3}\right)\cot\left(1 - \frac{x}{3}\right)\left(-\frac{1}{3}\right)$$

$$= \csc\left(1 - \frac{x}{3}\right)\cot\left(1 - \frac{x}{3}\right),$$

$$dy = \csc\left(1 - \frac{x}{3}\right)\cot\left(1 - \frac{x}{3}\right)dx.$$

(b) At the given values,

$$dy = \csc\left(1 - \frac{1}{3}\right)\cot\left(1 - \frac{1}{3}\right)(0.1)$$

$$= 0.1 \csc\frac{2}{3}\cot\frac{2}{3}$$

$$\approx 0.205525$$

21. **(a)** $y + xy - x = 0$

$y(1+x) = x$

$y = \dfrac{x}{x+1}$

Since $\dfrac{dy}{dx} = \dfrac{(x+1)(1)-(x)(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$,

$dy = \dfrac{dx}{(x+1)^2}$.

(b) At the given values, $dy = \dfrac{0.01}{(0+1)^2} = 0.01$.

22. **(a)** $2y = x^2 - xy$

$2dy = 2xdx - xdy - ydx$

$dy(2+x) = (2x-y)dx$

$dy = \left(\dfrac{2x-y}{2+x}\right)dx$

(b) At the given values, and $y = 1$ from the original equation,

$dy = \left(\dfrac{2(2)-1}{2+2}\right)(-0.05) = -0.0375$

23. $\dfrac{dy}{dx} = \sqrt{1-x^2}$

$dy = \left(-\dfrac{2x}{2\sqrt{1-x^2}}\right)dx$

$dy = -\dfrac{x}{\sqrt{1-x^2}}dx$

24. $\dfrac{dy}{dx} = e^{5x} + x^5$

$dy = (5e^{5x} + 5x^4)dx$

25. $\dfrac{d}{dx}\tan^{-1} u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$

$u = 4x$

$\dfrac{du}{dx} = 4$

$dy = \left(\dfrac{4}{1+16x^2}\right)dx$

26. $\dfrac{d}{dx}a^x = (\ln a)a^x$

$dy = (8^x \ln 8 + 8x^7)dx$

27. **(a)** $\Delta f = f(0.1) - f(0) = 0.21 - 0 = 0.21$

(b) Since $f'(x) = 2x+2$, $f'(0) = 2$. Therefore, $df = 2\,dx = 2(0.1) = 0.2$.

(c) $|\Delta f - df| = |0.21 - 0.2| = 0.01$

28. **(a)** $\Delta f = f(1.1) - f(1) = 0.231 - 0 = 0.231$

(b) Since $f'(x) = 3x^2 - 1$, $f'(1) = 2$. Therefore, $df = 2dx = 2(0.1) = 0.2$.

(c) $|\Delta f - df| = |0.231 - 0.2| = 0.031$

29. **(a)** $\Delta f = f(0.55) - f(0.5) = \dfrac{20}{11} - 2 = -\dfrac{2}{11}$

(b) Since $f'(x) = -x^{-2}$, $f'(0.5) = -4$. Therefore,

$df = -4\,dx = -4(0.05) = -0.2 = -\dfrac{1}{5}$

(c) $|\Delta f - df| = \left|-\dfrac{2}{11} + \dfrac{1}{5}\right| = \dfrac{1}{55}$

30. **(a)** $\Delta f = f(1.01) - f(1)$

$= 1.04060401 - 1$

$= 0.04060401$

(b) Since $f'(x) = 4x^3$, $f'(1) = 4$. Therefore, $df = 4\,dx = 4(0.01) = 0.04$.

(c) $|\Delta f - df| = |0.04060401 - 0.04|$

$= 0.00060401$

31. Note that $\dfrac{dV}{dr} = 4\pi r^2$, $dV = 4\pi r^2 dr$. When r changes from a to $a + dr$, the change in volume is approximately $4\pi a^2 dr$. When $a = 10$ and $dr = 0.05$,

$\Delta V \approx 4\pi(10)^2(0.05) = 20\pi \text{ cm}^3$.

32. Note that $\dfrac{dS}{dr} = 8\pi r$, so $dS = 8\pi r\,dr$. When r changes from a to $a + dr$, the change in surface area is approximately $8\pi a\,dr$. When $a = 10$ and $dr = 0.05$,

$\Delta S = 8\pi(10)(0.05) = 4\pi \text{ cm}^2$.

33. Note that $\frac{dV}{dx} = 3x^2$, so $dV = 3x^2\ dx$. When x changes from a to $a + dx$, the change in volume is approximately $3a^2\ dx$. When $a = 10$ and $dx = 0.05$, $\Delta V \approx 3(10)^2(0.05) = 15 \text{ cm}^3$.

34. Note that $\frac{dS}{dx} = 12x$, so $dS = 12x\ dx$. When x changes from a to $a + dx$, the change in surface area is approximately $12a\ dx$. When $a = 10$ and $dx = 0.05$, $\Delta S \approx 12(10)(0.05) = 6 \text{ cm}^2$.

35. Note that $\frac{dV}{dr} = 2\pi rh$, so $dV = 2\pi rh\ dr$. When r changes from a to $a + dr$, the change in volume is approximately $2\pi ah\ dr$. When $a = 10$ and $dr = 0.05$, $\Delta V \approx 2\pi(10)h(0.05) = \pi h \text{ cm}^3$.

36. Note that $\frac{dS}{dh} = 2\pi r$, so $dS = 2\pi r\ dh$. When h changes from a to $a + dh$, the change in lateral surface area is approximately $2\pi r\ dh$. When $a = 10$ and $dh = 0.05$, $\Delta S \approx 2\pi r(0.05) = 0.1\pi r \text{ cm}^2$.

37. $A = \pi r^2$
$dA = 2\pi r\, dr$
$dA = 2\pi(10)(0.1) \approx 6.3 \text{ in}^2$

38. $V = \frac{4}{3}\pi r^3$
$dV = 4\pi r^2 dr$
$dV = 4\pi(8)^2(0.3) \approx 241 \text{ in}^3$

39. $V = s^3$
$dV = 3s^2 ds$
$dV = 3(15)^2(0.2) = 135 \text{ cm}^3$

40. $A = \frac{\sqrt{3}}{4}s^2$
$dA = \frac{\sqrt{3}}{2}s\, ds$
$dA = \frac{\sqrt{3}}{2}(20)(0.5) = 8.7 \text{ cm}^2$

41. **(a)** Note that $f'(0) = \cos 0 = 1$.
$L(x) = f(0) + f'(0)(x-0) = 1 + 1x = x + 1$

(b) $f(0.1) \approx L(0.1) = 1.1$

(c) The actual value is less than 1.1. This is because the derivative is decreasing over the interval [0, 0.1], which means that the graph of $f(x)$ is concave down and lies below its linearization in this interval.

42. **(a)** Note that $A = \pi r^2$ and $\frac{dA}{dr} = 2\pi r$, so $dA = 2\pi r\ dr$. When r changes from a to $a + dr$, the change in area is approximately $2\pi a\ dr$. Substituting 2 for a and 0.02 for dr, the change in area is approximately
$2\pi(2)(0.02) = 0.08\pi \approx 0.2513$

(b) $\frac{dA}{A} = \frac{0.08\pi}{4\pi} = 0.02 = 2\%$

43. Let A = cross section area, C = circumference, and$D$ = diameter. Then $D = \frac{C}{\pi}$, so $\frac{dD}{dC} = \frac{1}{\pi}$ and $dD = \frac{1}{\pi}dC$. Also,
$A = \pi\left(\frac{D}{2}\right)^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$, so $\frac{dA}{dC} = \frac{C}{2\pi}$
and $dA = \frac{C}{2\pi}dC$. When C increases from 10π in. to $10\pi + 2$ in. the diameter increases by $dD = \frac{1}{\pi}(2) = \frac{2}{\pi} \approx 0.6366$ in. and the area increases by approximately
$dA = \frac{10\pi}{2\pi}(2) = 10 \text{ in}^2$.

44. Let x = edge length and V = volume. Then $V = x^3$, and so $dV = 3x^2\ dx$. With $x = 10$ cm and $dx = 0.01x = 0.1$ cm, we have
$V = 10^3 = 1000 \text{ cm}^3$ and
$dV = 3(10)^2(0.1) = 30 \text{ cm}^3$, so the percentage error in the volume measurement is approximately $\frac{dV}{V} = \frac{30}{1000} = 0.03 = 3\%$.

45. Let x = side length and A = area. Then $A = x^2$ and $\frac{dA}{dx} = 2x$, so $dA = 2x\,dx$. We want $|dA| \le 0.02A$, which gives $|2x\,dx| \le 0.02x^2$, or $|dx| \le 0.01x$. The side length should be measured with an error of no more than 1%.

46. **(a)** Note that $V = \pi r^2 h = 10\pi r^2 = 2.5\pi D^2$, where D is the interior diameter of the tank. Then $\frac{dV}{dD} = 5\pi D$, so $dV = 5\pi D\,dD$. We want $|dV| \le 0.01V$, which gives $|5\pi D\,dD| \le 0.01(2.5\pi D^2)$, or $|dD| \le 0.005D$. The interior diameter should be measured with an error of no more than 0.5%.

(b) Now we let D represent the *exterior* diameter of the tank, and we assume that the paint coverage rate (number of square feet covered per gallon of paint) is known precisely. Then, to determine the amount of paint within 5%, we need to calculate the lateral surface area S with an error of no more than 5%. Note that $S = 2\pi rh = 10\pi D$, so $\frac{dS}{dD} = 10\pi$ and $dS = 10\pi$. We want $|dS| \le 0.05S$, which gives $|10\pi\,dD| \le 0.05(10\pi D)$, or $dD \le 0.5D$. The exterior diameter should be measured with an error of no more than 5%.

47. Note that $V = \pi r^2 h$, where h is constant. Then $\frac{dV}{dr} = 2\pi rh$. The percent change is given by $\frac{dV}{V} = \frac{2\pi rhdr}{\pi r^2 h} = 2\frac{dr}{r} = 2\frac{0.1\%r}{r} = 0.2\%$.

48. Note that $\frac{dV}{dh} = 3\pi h^2$, so $dV = 3\pi h^2\,dh$. We want $|dV| \le 0.01V$, which gives $|3\pi h^2 dh| \le 0.01(\pi h^3)$, or $|dh| \le \frac{0.01h}{3}$. The height should be measured with an error of no more than $\frac{1}{3}\%$.

49. If $dC = 2\pi\,dr$ and $dC = \frac{1}{8}$ inch, then $dr = \frac{1}{16\pi}$ inch. Since $V = \frac{4}{3}\pi r^3$, we have $dV = 4\pi r^2 dr = 4\pi r^2\left(\frac{1}{16\pi}\right) = \frac{r^2}{4}$.

The volume error in each case is simply $\frac{r^2}{4}\text{in}^3$.

Sphere Type	True Radius	Tape error	Radius Error	Volume Error
Orange	2 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	1in^3
Melon	4 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	4in^3
Beach Ball	7 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in..	12.25 in^3

50. If $dC = 2\pi\,dr$ and $dC = \frac{1}{8}$ inch, then $dr = \frac{1}{16\pi}$ inch. Since $A = 4\pi r^2$, we have $dA = 8\pi r dr = 8\pi r\left(\frac{1}{16\pi}\right) = \frac{r}{2}$.

The surface area error in each case is simply $\frac{r}{2}\text{ in}^2$.

Sphere Type	True Radius	Tape Error	Radius Error	Volume Error
Orange	2 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	1in^2
Melon	4 in.	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	2in^2
Beach Ball	7 in	$\frac{1}{8}$ in.	$\frac{1}{16\pi}$ in.	3.5in^2

51. We have $\frac{dW}{dg} = -bg^{-2}$, so $dW = -bg^{-2}dg$.

Then $\frac{dW_{\text{moon}}}{dW_{\text{earth}}} = \frac{-b(5.2)^{-2}dg}{-b(32)^{-2}dg} = \frac{32^2}{5.2^2} \approx 37.87$.

The ratio is about 37.87 to 1.

52. **(a)** Note that $T = 2\pi L^{1/2} g^{-1/2}$, so

$$\frac{dT}{dg} = -\pi L^{1/2} g^{-3/2} \text{ and}$$

$$dT = -\pi L^{1/2} g^{-3/2} dg.$$

(b) Note that dT and dg have opposite signs. Thus, if g increases, T decreases and the clock speeds up.

(c) $$-\pi L^{1/2} g^{-3/2} dg = dT$$

$$-\pi(100)^{1/2}(980)^{-3/2} dg = 0.001$$

$$dg \approx -0.9765$$

Since

$dg \approx -0.9765$, $g \approx 980 - 0.9765 = 979.0235$.

53. Let $f(x) = x^3 + x - 1$. Then $f'(x) = 3x^2 + 1$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}.$$

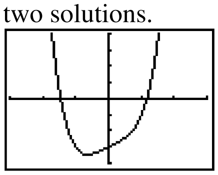
Note that f is cubic and f' is always positive, so there is exactly one solution. We choose $x_1 = 0$.

$x_1 = 0$
$x_2 = 1$
$x_3 = 0.75$
$x_4 \approx 0.6860465$
$x_5 \approx 0.6823396$
$x_6 \approx 0.6823278$
$x_7 \approx 0.6823278$
Solution: $x \approx 0.682328$.

54. Let $f(x) = x^4 + x - 3$. Then $f'(x) = 4x^3 + 1$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$$

The graph of $y = f(x)$ shows that $f(x) = 0$ has two solutions.



[−3, 3] by [−4, 4]

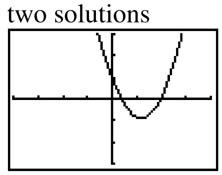
$x_1 = -1.5$	$x_1 = 1.2$
$x_2 = -1.455$	$x_2 \approx 1.6541962$
$x_3 \approx -1.4526332$	$x_3 \approx 1.1640373$
$x_4 \approx -1.4526269$	$x_4 \approx 1.1640351$
$x_5 \approx -1.4526269$	$x_5 \approx 1.1640351$

Solution: $x \approx -1.452627, 1.164035$

55. Let $f(x) = x^2 - 2x + 1 - \sin x$.
Then $f'(x) = 2x - 2 - \cos x$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2x_n + 1 - \sin x_n}{2x_n - 2 - \cos x_n}$$

The graph of $y = f(x)$ shows that $f(x) = 0$ has two solutions



[−4, 4] by [−3, 3]

$x_1 = 0.3$	$x_1 = 2$
$x_2 \approx 0.3825699$	$x_2 \approx 1.9624598$
$x_3 \approx 0.3862295$	$x_3 \approx 1.9615695$
$x_4 \approx 0.3862369$	$x_4 \approx 1.9615690$
$x_5 \approx 0.3862369$	$x_5 \approx 1.9615690$

Solutions: $x \approx 0.386237, 1.961569$

56. Let $f(x) = x^4 - 2$. Then $f'(x) = 4x^3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 2}{4x_n^3}.$$

Note that $f(x) = 0$ clearly has two solutions, namely $x = \pm\sqrt[4]{2}$. We use Newton's method to find the decimal equivalents.

$x_1 = 1.5$
$x_2 \approx 1.2731481$
$x_3 \approx 1.1971498$
$x_4 \approx 1.1892858$
$x_5 \approx 1.1892071$
$x_6 \approx 1.1892071$
Solutions: $x \approx \pm 1.189207$

57. True; a look at the graph reveals the problem. The graph decreases after $x = 1$ toward a horizontal asymptote of $y = 0$, so the x-intercepts of the tangent lines keep getting bigger without approaching a zero.

58. False; by the product rule, $d(uv) = u\,dv + v\,du$.

59. B; $f(x)=e^x$

$$f'(x)=e^x$$
$$L(x)=e^1+e^1(x-1)$$
$$L(x)=ex$$

60. D; $y=\tan x$

$$dy=(\sec^2 x)dx=(\sec^2 \pi)0.5=0.5$$

61. D; $f(x)=x-x^3+2$

$$f'(x)=1-3x^2$$
$$x_{n+1}=x_n-\frac{x_n-x_n^3+2}{1-3x_n^2}$$
$$x_2=1-\frac{1-(1)^3+2}{1-3(1)^2}=2$$
$$x_3=2-\frac{2-(2)^3+2}{1-3(2)^2}=\frac{18}{11}$$

62. A; $f(x)=\sqrt[3]{x};\ x=64$

$$f'(64)=\frac{1}{3}(64)^{-2/3}=\frac{1}{48}$$
$$\sqrt[3]{66}\approx 4+\frac{1}{48}(66-64)=\frac{97}{24}$$

The percentage error is

$$\frac{\sqrt[3]{66}-97/24}{\sqrt[3]{66}}\approx 0.01\%.$$

63. If $f'(x)\neq 0$, we have

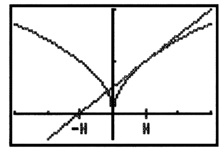
$x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}=x_1-\dfrac{0}{f'(x_1)}=x_1$. Therefore, $x_2=x_1$, and all later approximations are also equal to x_1.

64. If $x_1=h$, then $f'(x_1)=\dfrac{1}{2h^{1/2}}$ and

$x_2=h-\dfrac{h^{1/2}}{\frac{1}{2h^{1/2}}}=h-2h=-h$. If $x_1=-h$, then

$f'(x_1)=-\dfrac{1}{2\sqrt{h}}$ and

$$x_2=-h-\frac{h^{1/2}}{-\frac{1}{2h^{1/2}}}=-h+2h=h$$



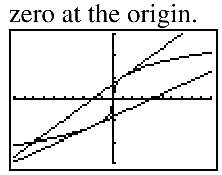
[−3, 3] by [−0.5, 2]

65. Note that $f'(x)=\dfrac{1}{3}x^{-2/3}$ and so

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$
$$=x_n-\frac{x_n^{1/3}}{\frac{x_n^{-2/3}}{3}}$$
$$=x_n-3x_n$$
$$=-2x_n.$$

For $x_1=1$, we have $x_2=-2,\ x_3=4,\ x_4=-8,$ and $x_5=16;\ |x_n|=2^{n-1}.$

The approximations alternate in sign and rapidly get farther and farther away from the zero at the origin.



[−10, 10] by [−3, 3]

66. **(a)** **i.** $Q(a)=f(a)$ implies that $b_0=f(a)$.

ii. Since $Q'(x)=b_1+2b_2(x-a)$, $Q'(a)=f'(a)$ implies that $b_1=f'(a)$.

iii. Since $Q''(x)=2b_2,\ Q''(a)=f''(a)$ implies that $b_2=\dfrac{f''(a)}{2}$

In summary,

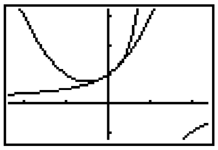
$b_0=f(a),\ b_1=f'(a),$ and $b_2=\dfrac{f''(a)}{2}.$

(b) $f(x)=(1-x)^{-1}$

$$f'(x)=-1(1-x)^{-2}(-1)=(1-x)^{-2}$$
$$f''(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}$$

Since $f(0)=1, f'(0)=1,$ and $f''(0)=2,$ the coefficients are $b_0=1,\ b_1=1,$ and $b_2=\dfrac{2}{2}=1.$ The quadratic approximation is $Q(x)=1+x+x^2.$

(c)



[–2.35, 2.35] by [–1.25, 3.25]

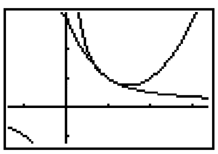
As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(d) $g(x) = x^{-1}$

$g'(x) = -x^{-2}$

$g''(x) = 2x^{-3}$

Since $g(1) = 1, g'(1) = -1,$ and $g''(1) = 2,$ the coefficients are $b_o = 1, b_1 = -1,$ and $b_2 = \frac{2}{2} = 1.$ The quadratic approximation is $Q(x) = 1 - (x-1) + (x-1)^2.$



[–1.35, 3.35] by [–1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical

(e) $h(x) = (1+x)^{1/2}$

$h'(x) = \frac{1}{2}(1+x)^{-1/2}$

$h''(x) = -\frac{1}{4}(1+x)^{-3/2}$

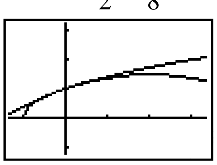
Since

$h(0) = 1, h'(0) = \frac{1}{2},$ and $h''(0) = -\frac{1}{4},$ the coefficients are $b_0 = 1,\ b_1 = \frac{1}{2},$ and

$b_2 = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}.$

The quadratic approximation is

$Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}.$



[–1.35, 3.35] by [–1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(f) The linearization of any differentiable function $u(x)$ at $x = a$ is $L(x) = u(a) + u'(a)(x-a) = b_0 + b_1(x-a),$ where b_0 and b_1 are the coefficients of the constant and linear terms of the quadratic approximation. Thus, the linearization for $f(x)$ at $x = 0$ is $1 + x$; the linearization for $g(x)$ at $x = 1$ is $1 - (x-1)$ or $2 - x$; and the linearization for $h(x)$ at $x = 0$ is $1 + \frac{x}{2}.$

67. Finding a zero of $\sin x$ by Newton's method would use the recursive formula $x_{n+1} = x_n - \frac{\sin(x_n)}{\cos(x_n)} = x_n - \tan x_n,$ and that is exactly what the calculator would be doing. Any zero of $\sin x$ would be a multiple of π.

68. Just multiply the corresponding derivative formulas by dx.

(a) Since $\frac{d}{dx}(c) = 0,\ d(c) = 0.$

(b) Since $\frac{d}{dx}(cu) = c\frac{du}{dx},\ d(cu) = c\ du.$

(c) Since $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx},$
$d(u+v) = du + dv.$

(d) Since $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx},$
$d(u \cdot v) = u\ dv + v\ du.$

(e) Since $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2},$
$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$

(f) Since $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx},$
$d(u^n) = nu^{n-1}du.$

69. First, note that $\frac{d}{dx}\tan x = \sec^2 x,$ and $\sec^2(0) = 1,$ so the linearization of $\tan x$ at $x = 0$ is $y - 0 = 1(x - 0),$ or more simply $y = x.$

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x/\cos x}{x}\\ &= \lim_{x\to 0}\left(\frac{1}{\cos x}\frac{\sin x}{x}\right)\\ &= \left(\lim_{x\to 0}\frac{1}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\\ &= (1)(1)\\ &= 1.\end{aligned}$$

70. $g(a) = c$, so if $E(a) = 0$, then $g(a) = f(a)$ and $c = f(a)$. Then
$E(x) = f(x) - g(x) = f(x) - f(a) - m(x - a).$
Thus, $\frac{E(x)}{x-a} = \frac{f(x)-f(a)}{x-a} - m.$

$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a),$ so

$\lim_{x\to a}\frac{E(x)}{x-a} = f'(a) - m.$

Therefore, if the limit of $\frac{E(x)}{x-a}$ is zero, then $m = f'(a)$ and $g(x) = L(x).$

71. $f'(x) = \frac{1}{2\sqrt{x+1}} + \cos x$

We have $f(0) = 1$ and $f'(0) = \frac{3}{2}$

$$\begin{aligned}L(x) &= f(0) + f'(0)(x-0)\\ &= 1 + \frac{3}{2}x\end{aligned}$$

The linearization is the sum of the two individual linearizations, which are x for $\sin x$ and $1 + \frac{1}{2}x$ for $\sqrt{x+1}.$

72. The equation for the tangent is $y - f(x_n) = f'(x_n)(x - x_n).$ Set $y = 0$ and solve for x.

$$\begin{aligned}0 - f(x_n) &= f'(x_n)(x - x_n)\\ -f(x_n) &= f'(x_n)\cdot x - f'(x_n)\cdot x_n\\ f'(x_n)\cdot x &= f'(x_n)\cdot x_n - f(x_n)\\ x &= x_n - \frac{f(x_n)}{f'(x_n)} \quad (\text{If } f'(x_n) \neq 0)\end{aligned}$$

The value of x is the next approximation x_{n+1}.

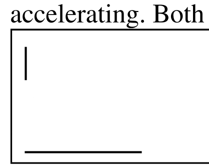
Section 5.6 Related Rates (pp. 250–259)

Exploration 1 The Sliding Ladder

1. Here the x-axis represents the ground and the y-axis represents the wall. The curve (x_1, y_1) gives the position of the bottom of the ladder (distance from the wall) at any time t in $0 \le t \le 5.$ The curve (x_2, y_2) gives the position of the top of the ladder at any time in $0 \le t \le 5.$

2. $0 \le t \le 5$

4. This is a snapshot at $t \approx 3.1$. The top of the ladder is moving down the y-axis and the bottom of the ladder is moving to the right on the x-axis. The end of the ladder is accelerating. Both axes are hidden from view.



[–1, 15] by [–1, 15]

6. $\frac{dy}{dt} = \frac{-4T}{\sqrt{10^2 - (2T)^2}}$

7. $y'(3) = -1.5\text{ ft/sec}^2$. The negative number means the y-side of the right triangle is decreasing in length.

8. Since $\lim_{t\to 5^-} y'(t) = -\infty,$ the speed of the top of the ladder is infinite as it hits the ground.

Quick Review 5.6

1. $D = \sqrt{(7-0)^2 + (0-5)^2} = \sqrt{49+25} = \sqrt{74}$

2. $D = \sqrt{(b-0)^2 + (0-a)^2} = \sqrt{a^2+b^2}$

3. Use implicit differentiation.

$$\begin{aligned}\frac{d}{dx}(2xy + y^2) &= \frac{d}{dx}(x+y)\\ 2x\frac{dy}{dx} + 2y(1) + 2y\frac{dy}{dx} &= (1) + \frac{dy}{dx}\\ (2x+2y-1)\frac{dy}{dx} &= 1 - 2y\\ \frac{dy}{dx} &= \frac{1-2y}{2x+2y-1}\end{aligned}$$

4. Use implicit differentiation.

$$\frac{d}{dx}(x\sin y) = \frac{d}{dx}(1-xy)$$
$$(x)(\cos y)\frac{dy}{dx} + (\sin y)(1) = -x\frac{dy}{dx} - y(1)$$
$$(x + x\cos y)\frac{dy}{dx} = -y - \sin y$$
$$\frac{dy}{dx} = \frac{-y-\sin y}{x + x\cos y}$$
$$\frac{dy}{dx} = -\frac{y+\sin y}{x + x\cos y}$$

5. Use implicit differentiation.

$$\frac{d}{dx}x^2 = \frac{d}{dx}\tan y$$
$$2x = \sec^2 y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2x}{\sec^2 y}$$
$$\frac{dy}{dx} = 2x\cos^2 y$$

6. Use implicit differentiation.

$$\frac{d}{dx}\ln(x+y) = \frac{d}{dx}(2x)$$
$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right) = 2$$
$$1+\frac{dy}{dx} = 2(x+y)$$
$$\frac{dy}{dx} = 2x + 2y - 1$$

7. Using $A(-2, 1)$ we create the parametric equations $x = -2 + at$ and $y = 1 + bt$, which determine a line passing through A *at* $t = 0$. We determine a and b so that the line passes through $B(4, -3)$ *at* $t = 1$. Since $4 = -2 + a$, we have $a = 6$, and since $-3 = 1 + b$, we have $b = -4$. Thus, one parametrization for the line segment is $x = -2 + 6t$, $y = 1 - 4t$, $0 \le t \le 1$. (Other answers are possible.)

8. Using $A(0, -4)$, we create the parametric equations $x = 0 + at$ and $y = -4 + bt$, which determine a line passing through A at $t = 0$. We now determine a and b so that the line passes through $B(5, 0)$ at $t = 1$. Since $5 = 0 + a$, we have $a = 5$, and since $0 = -4 + b$, we have $b = 4$. Thus, one parametrization for the line segment is $x = 5t$, $y = -4 + 4t$, $0 \le t \le 1$. (Other answers are possible.)

9. One possible answer: $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

10. One possible answer: $\frac{3\pi}{2} \le t \le 2\pi$

Section 5.6 Exercises

1. Since $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$, we have $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$.

2. Since $\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$, we have $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$.

3. (a) Since $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$, we have

$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt}.$$

(b) Since $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$, we have

$$\frac{dV}{dt} = 2\pi rh\frac{dr}{dt}.$$

(c)
$$\frac{dV}{dt} = \frac{d}{dt}\pi r^2 h = \pi\frac{d}{dt}(r^2 h)$$
$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + h(2r)\frac{dr}{dt}\right)$$
$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$$

4. (a)
$$\frac{dP}{dt} = \frac{d}{dt}(RI^2)$$
$$\frac{dP}{dt} = R\frac{d}{dt}I^2 + I^2\frac{dR}{dt}$$
$$\frac{dP}{dt} = R\left(2I\frac{dI}{dt}\right) + I^2\frac{dR}{dt}$$
$$\frac{dP}{dt} = 2RI\frac{dI}{dt} + I^2\frac{dR}{dt}$$

(b) If P is constant, we have $\frac{dP}{dt} = 0$, which means $2RI\frac{dI}{dt} + I^2\frac{dR}{dt} = 0$, or

$$\frac{dR}{dt} = -\frac{2R}{I}\frac{dI}{dt} = -\frac{2P}{I^3}\frac{dI}{dt}.$$

5. $\frac{ds}{dt} = \frac{d}{dt}\sqrt{x^2 + y^2 + z^2}$

$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \frac{d}{dt}(x^2 + y^2 + z^2)$

$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}\right)$

$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}}$

6. $\frac{dA}{dt} = \frac{d}{dt}\left(\frac{1}{2}ab\sin\theta\right)$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{da}{dt}\cdot b\cdot\sin\theta + a\cdot\frac{db}{dt}\cdot\sin\theta + ab\cdot\frac{d}{dt}\sin\theta\right)$

$\frac{dA}{dt} = \frac{1}{2}\left(b\sin\theta\frac{da}{dt} + a\sin\theta\frac{db}{dt} + ab\cos\theta\frac{d\theta}{dt}\right)$

7. **(a)** Since V is increasing at the rate of 1 volt/sec, $\frac{dV}{dt} = 1$ volt/sec.

(b) Since I is decreasing at the rate of $\frac{1}{3}$ amp/sec, $\frac{dI}{dt} = -\frac{1}{3}$ amp/sec.

(c) Differentiating both sides of $V = IR$, we have $\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$.

(d) Note that $V = IR$ gives $12 = 2R$, so $R = 6$ ohms. Now substitute the known values into the equation in (c).

$1 = 2\frac{dR}{dt} + 6\left(-\frac{1}{3}\right)$

$3 = 2\frac{dR}{dt}$

$\frac{dR}{dt} = \frac{3}{2}$ ohms/sec

R is changing at the rate of $\frac{3}{2}$ ohms/sec. Since this value is positive, R is increasing.

8. Step 1:
r = radius of plate
A = area of plate
Step 2:
At the instant in question, $\frac{dr}{dt} = 0.01$ cm/sec, $r = 50$ cm.
Step 3:
We want to find $\frac{dA}{dt}$.
Step 4:
$A = \pi r^2$

Step 5:
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$
Step 6:
$$\frac{dA}{dt} = 2\pi(50)(0.01) = \pi \text{ cm}^2/\text{sec}$$
At the instant in question, the area is increasing at the rate of $\pi \text{cm}^2/\text{sec}$.

9. Step 1:
l = length of rectangle
w = width of rectangle
A = area of rectangle
P = perimeter of rectangle
D = length of a diagonal of the rectangle
Step 2:
At the instant in question, $\frac{dl}{dt} = -2$ cm/sec, $\frac{dw}{dt} = 2$ cm/sec, $l = 12$ cm, and $w = 5$ cm.
Step 3:
We want to find $\frac{dA}{dt}, \frac{dP}{dt}$, and $\frac{dD}{dt}$.
Steps 4, 5, and 6:

(a) $A = lw$
$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt}$$
$$\frac{dA}{dt} = (12)(2) + (5)(-2) = 14 \text{ cm}^2/\text{sec}$$
The rate of change of the area is $14 \text{ cm}^2/\text{sec}$.

(b) $P = 2l + 2w$
$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$
$$\frac{dP}{dt} = 2(-2) + 2(2) = 0 \text{ cm/sec}$$
The rate of change of the perimeter is 0 cm/sec.

(c) $D = \sqrt{l^2 + w^2}$
$$\frac{dD}{dt} = \frac{1}{2\sqrt{l^2+w^2}}\left(2l\frac{dl}{dt} + 2w\frac{dw}{dt}\right) = \frac{l\frac{dl}{dt} + w\frac{dw}{dt}}{\sqrt{l^2+w^2}}$$
$$\frac{dD}{dt} = \frac{(12)(-2)+(5)(2)}{\sqrt{12^2+5^2}} = -\frac{14}{13} \text{ cm/sec}$$
The rate of change of the length of the diameter is $-\frac{14}{13}$ cm/sec.

(d) The area is increasing, because its derivative is positive. The perimeter is not changing, because its derivative is zero. The diagonal length is decreasing, because its derivative is negative.

10. Step 1:
x, y, z = edge lengths of the box
V = volume of the box
S = surface area of the box
s = diagonal length of the box
Step 2:
At the instant in question,
$\frac{dx}{dt} = 1$ m/sec, $\frac{dy}{dt} = -2$ m/sec, $\frac{dz}{dt} = 1$ m/sec, $x = 4$m, $y = 3$m, and $z = 2$m.
Step 3:
We want to find $\frac{dV}{dt}, \frac{dS}{dt}$, and $\frac{ds}{dt}$.
Steps 4, 5, and 6:

(a) $V = xyz$
$$\frac{dV}{dt} = xy\frac{dz}{dt} + xz\frac{dy}{dt} + yz\frac{dx}{dt}$$
$$\frac{dV}{dt} = (4)(3)(1) + (4)(2)(-2) + (3)(2)(1) = 2 \text{ m}^3/\text{sec}$$
The rate of change of the volume is $2 \text{ m}^3/\text{sec}$.

(b) $S = 2(xy + xz + yz)\ \pi$
$$\frac{dS}{dt}\ 2\left(x\frac{dy}{dt} + y\frac{dx}{dt} + x\frac{dz}{dt} + z\frac{dx}{dt} + y\frac{dz}{dt} + z\frac{dy}{dt}\right)$$
$$\frac{dS}{dt} = 2[(4)(-2) + (3)(1) + (4)(1) + (2)(1) + (3)(1) + (2)(-2)] = 0 \text{ m}^2/\text{sec}$$
The rate of change of the surface area is $0 \text{ m}^2/\text{sec}$.

(c) $s = \sqrt{x^2 + y^2 + z^2}$

$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dy}\right)$$

$$= \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}}$$

The rate of change of the diagonal length is 0 m/sec.

11. Step 1:
r = radius of spherical balloon
S = surface area of spherical balloon
V = volume of spherical balloon
Step 2:
At the instant in question, $\frac{dV}{dt} = 100\pi \text{ ft}^3\text{/min}$ and $r = 5$ ft.
Step 3:
We want to find the values of $\frac{dr}{dt}$ and $\frac{dS}{dt}$.
Steps 4, 5, and 6:

(a) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$100\pi = 4\pi(5)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = 1 \text{ ft/min}$$

The radius is increasing at the rate of 1 ft/min.

(b) $S = 4\pi r^2$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi(5)(1)$$

$$\frac{dS}{dt} = 40\pi \text{ ft}^2\text{ / min}$$

The surface area is increasing at the rate of 40π ft^2/min.

12. Step 1:
r = radius of spherical droplet
S = surface area of spherical droplet
V = volume of spherical droplet
Step 2:
No numerical information is given.
Step 3:
We want to show that $\frac{dr}{dt}$ is constant.
Step 4:
$S = 4\pi r^2, V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = kS$ for some constant k
Steps 5 and 6:
Differentiating $V = \frac{4}{3}\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Substituting kS for $\frac{dV}{dt}$ and S for $4\pi r^2$, we have $kS = S\frac{dr}{dt}$, or $\frac{dr}{dt} = k$.

13. Step 1:
s = (diagonal) distance from antenna to airplane
x = horizontal distance from antenna to airplane
Step 2:
At the instant in question,
$s = 10$ mi and $\frac{ds}{dt} = 300$ mph.
Step 3:
We want to find $\frac{dx}{dt}$.
Step 4:
$x^2 + 49 = s^2$ or $x = \sqrt{s^2 - 49}$
Step 5:

$$\frac{dx}{dt} = \frac{1}{2\sqrt{s^2 - 49}}\left(2s\frac{ds}{dt}\right) = \frac{s}{\sqrt{s^2 - 49}}\frac{ds}{dt}$$

Step 6:

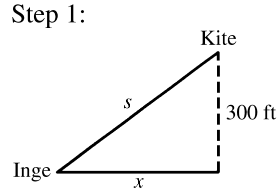
$$\frac{dx}{dt} = \frac{10}{\sqrt{10^2 - 49}}(300)$$

$$= \frac{3000}{\sqrt{51}} \text{ mph}$$

$$\approx 420.08 \text{ mph}$$

The speed of the airplane is about 420.08 mph.

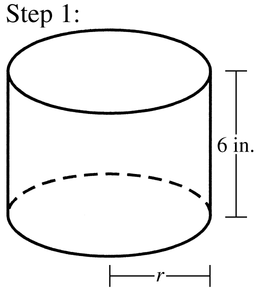
14. Step 1:



s = length of kite string
x = horizontal distance from Inge to kite

Step 2:
At the instant in question, $\frac{dx}{dt} = 25$ ft/sec and $s = 500$ ft
Step 3:
We want to find $\frac{ds}{dt}$.
Step 4:
$x^2 + 300^2 = s^2$
Step 5:
$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$ or $x\frac{dx}{dt} = s\frac{ds}{dt}$
Step 6:
At the instant in question, since $x^2 + 300^2 = s^2$, we have
$x = \sqrt{s^2 - 300^2} = \sqrt{500^2 - 300^2} = 400.$
Thus $(400)(25) = (500)\frac{ds}{dt}$, so $\frac{ds}{dt}$, so $\frac{ds}{dt} = 20$ ft/sec. Inge must let the string out at the rate of 20 ft/sec.

15. Step 1:

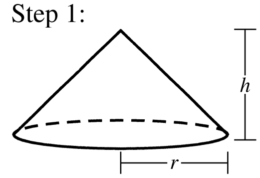


The cylinder shown represents the shape of the hole.
r = radius of cylinder
V = volume of cylinder
Step 2:
At the instant in question,
$\frac{dr}{dt} = \frac{0.001 \text{ in.}}{3 \text{ min}} = \frac{1}{3000}$ in./min and (since the diameter is 3.800 in.), $r = 1.900$ in.
Step 3:
We want to find $\frac{dV}{dt}$.
Step 4:
$V = \pi r^2(6) = 6\pi r^2$
Step 5:
$\frac{dV}{dt} = 12\pi r\frac{dr}{dt}$
Step 6:

$$\begin{aligned}\frac{dV}{dt} &= 12\pi(1.900)\left(\frac{1}{3000}\right)\\ &= \frac{19\pi}{2500}\\ &= 0.0076\pi\\ &\approx 0.0239 \text{ in}^3/\text{min.}\end{aligned}$$

The volume is increasing at the rate of approximately $0.0239 \text{ in}^3/\text{min}$.

16. Step 1:



r = base radius of cone
h = height of cone
V = volume of cone
Step 2:
At the instant in question, $h = 4$ m and $\frac{dV}{dt} = 10 \text{ m}^3/\text{min}$.
Step 3:
We want to find $\frac{dh}{dt}$ and $\frac{dr}{dt}$.
Step 4:
Since the height is $\frac{3}{8}$ of the base diameter, we have $h = \frac{3}{8}(2r)$ or $r = \frac{4}{3}h$.
We also have
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{4}{3}h\right)^2 h = \frac{16\pi h^3}{27}$. We will use the equations $V = \frac{16\pi h^3}{27}$ and $r = \frac{4}{3}h$.
Step 5 and 6:

(a) $\frac{dV}{dt} = \frac{16\pi h^2}{9}\frac{dh}{dt}$

$$10 = \frac{16\pi(4)^2}{9}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{45}{128\pi} \text{ m/min} = \frac{1125}{32\pi} \text{ cm/min}$$

The height is changing at the rate of $\frac{1125}{32\pi} \approx 11.19$ cm/min.

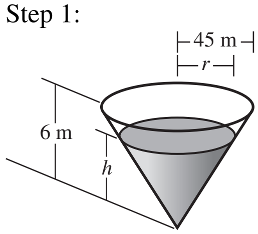
(b) Using the results from Step 4 and part (a), we have

$$\frac{dr}{dt} = \frac{4}{3}\frac{dh}{dt} = \frac{4}{3}\left(\frac{1125}{32\pi}\right) = \frac{375}{8\pi}\text{cm/min}.$$

The radius is changing at the rate of

$$\frac{375}{8\pi} \approx 14.92\,\text{cm/min}.$$

17. Step 1:



r = radius of top surface of water
h = depth of water in reservoir
V = volume of water in reservoir
Step 2:
At the instant in question,

$$\frac{dV}{dt} = -50\ \text{m}^3/\text{min and } h = 5\ \text{m}.$$

Step 3:

We want to find $-\frac{dh}{dt}$ and $\frac{dr}{dt}$.

Step 4:

Note that $\frac{h}{r} = \frac{6}{45}$ by similar cones, so $r = 7.5h$.

Then $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(7.5h)^2 h = 18.75\pi h^3$

Steps 5 and 6:

(a) Since $V = 18.75\pi h^3, \frac{dV}{dt} = 56.25\pi h^2 \frac{dh}{dt}$.

Thus $-50 = 56.25\pi(5^2)\frac{dh}{dt}$, and

so $\frac{dh}{dt} = -\frac{8}{225\pi}$ m/min $= -\frac{32}{9\pi}$ cm/min.

The water level is falling by

$$\frac{32}{9\pi} \approx 1.13\,\text{cm/min}.$$

(Since $\frac{dh}{dt} < 0$, the rate at which the water level is *falling* is positive.)

(b) Since $r = 7.5h$,

$\frac{dr}{dt} = 7.5\frac{dh}{dt} = -\frac{80}{3\pi}$ cm/min. The rate of change of the radius of the water's surface is $-\frac{80}{3\pi} \approx -8.49$ cm/min.

18. **(a)** Step 1:
y = depth of water in bowl
V = volume of water in bowl
Step 2:
At the instant in question,

$$\frac{dV}{dt} = -6\ \text{m}^3/\text{min and } y = 8\ \text{m}.$$

Step 3:

We want to find the value of $\frac{dy}{dt}$.

Step 4:

$$V = \frac{\pi}{3}y^2(39 - y) \text{ or } V = 13\pi y^2 - \frac{\pi}{3}y^3$$

Step 5:

$$\frac{dV}{dt} = (26\pi y - \pi y^2)\frac{dy}{dt}$$

Step 6:

$$-6 = \left[26\pi(8) - \pi(8^2)\right]\frac{dy}{dt}$$

$$-6 = 144\pi\frac{dy}{dt}$$

$$\frac{dy}{dt} = -\frac{1}{24\pi} \approx -0.01326 \text{ m/min}$$

or $-\frac{25}{6\pi} \approx -1.326$ cm/min

(b) Since $r^2 + (13 - y)^2 = 13^2$,

$$r = \sqrt{169 - (13 - y)^2} = \sqrt{26y - y^2}.$$

(c) Step 1:
y = depth of water
r = radius of water surface
V = volume of water in bowl
Step 2:
At the instant in question,

$$\frac{dV}{dt} = -6\ \ \text{m}^3/\text{min},\ \ y = 8\,\text{m, and}$$

therefore (from part (a))

$$\frac{dy}{dt} = -\frac{1}{24\pi}\ \text{m/min}.$$